Conditional Distributions and Independent random variables

1. Conditional Distributions

Definition 1. *Let $\underline{Z} = (X, Y)$ be a random vector of discrete type with support $E_{\underline{Z}}$, joint d.f. $F_{\underline{Z}}$ and joint p.m.f. $f_{\underline{Z}}$. Then X and Y are discrete type random variables.*

For a fixed y with $P(Y = y) > 0$, the function $f_{X|Y}(.|y) : \mathbb{R} \longrightarrow \mathbb{R}$ defined as

$$f_{X|Y}(x|y) = P(X = x|Y = y), \ \forall \ x \in \mathbb{R},$$

is called the conditional probability mass function of X, given $Y = y$. Thus, the conditional probability mass function of X, given $Y = y$, is

$$f_{X|Y}(x|y) = P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{f_{\underline{Z}}(x, y)}{f_Y(y)}$$
$$= \begin{cases} \frac{f_{\underline{Z}}(x,y)}{f_Y(y)}, & \textit{if } x \in E_{X|Y=y} \\ 0, & \textit{otherwise,} \end{cases}$$

where $E_{X|Y=y} = \{x \in \mathbb{R} \ | \ (x, y) \in E_{\underline{Z}}\}$ and f_Y is the marginal p.m.f. of Y.

The conditional cumulative distribution function of X, given $Y = y$, is defined as

$$\begin{aligned} F_{X|Y}(x|y) &= P(X \leq x|Y = y) \\ &= \frac{P(X \leq x, Y = y)}{P(Y = y)} \\ &= \sum_{x_i \in E_{X|Y=y} \cap (-\infty, x]} \frac{f_{\underline{Z}}(x_i, y)}{f_Y(y)} \\ &= \sum_{x_i \leq x} f_{X|Y}(x_i|y), \ \textit{where } x_i \in E_{X|Y=y}. \end{aligned}$$

In the similar manner, we can define the conditional probability mass function and conditional cumulative distribution function of Y, given $X = x$, provided $P(X = x) > 0$.

Definition 2. *Let $\underline{Z} = (X, Y)$ be a random vector of continuous type with joint c.d.f. $F_{\underline{Z}}$ and joint p.d.f. $f_{\underline{Z}}$. Then X and Y are continuous type random variables. Let $y \in \mathbb{R}$ be such that $f_Y(y) > 0$, where $f_Y(y) > 0$ is the marginal p.d.f. of Y.*

The function $f_{X|Y}(.|y) : \mathbb{R} \longrightarrow \mathbb{R}$ defined as

$$f_{X|Y}(x|y) = \frac{f_{\underline{Z}}(x, y)}{f_Y(y)}, \ \forall \ x \in \mathbb{R},$$

is called the conditional probability density function of X, given $Y = y$.

Also, the conditional cumulative distribution function of X, given $Y = y$, is defined as

$$\begin{aligned} F_{X|Y}(x|y) &= \int_{-\infty}^{x} f_{X|Y}(t|y) dt \\ &= \int_{-\infty}^{x} \frac{f_{\underline{Z}}(t, y)}{f_Y(y)} dt \end{aligned}$$

In the similar manner, we can define the conditional probability density function and conditional cumulative distribution function of Y, given $\{X = x\}$, provided $f_X(x) > 0$, where $f_X(x) > 0$ is the marginal p.d.f. of X.

Note: Definition 1 and 2 can be generalized if we replace random variables X and Y by random vectors $\underline{X}$ and $\underline{Y}$.

Example 3. *Let $\underline{Z} = (X, Y)$ be a random vector with joint p.d.f.*

$$f(x,y) = \begin{cases} 6xy(2-x-y), & \text{if } 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then find the conditional p.d.f. of X, given $Y = y$, where $0 < y < 1$.

Solution: The conditional p.d.f. of X, given $Y = y$, is

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \begin{cases} \frac{6xy(2-x-y)}{\int_0^1 6xy(2-x-y)dx}, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{6x(2-x-y)}{4-3y}, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Example 4. *Let $\underline{Z} = (X, Y, Z)$ be a random vector with joint p.m.f.*

$$f(x,y,z) = \begin{cases} \frac{xyz}{72}, & \text{if } (x,y,z) \in \{1,2\} \times \{1,2,3\} \times \{1,3\} \\ 0, & \text{otherwise} \end{cases}$$

(1) *Find the conditional p.m.f. of X, given $(Y, Z) = (2, 1)$.*
(2) *Find the conditional p.m.f. of (X, Z), given $Y = 3$.*

Solution:

(1) The conditional p.m.f. of X, given $(Y, Z) = (2, 1)$, is

$$\begin{aligned} f_{X|(Y,Z)}(x|(2,1)) &= \frac{f(x,2,1)}{P((Y,Z) = (2,1))} \\ &= \begin{cases} \frac{2x}{72P(Y=2,Z=1)}, & \text{if } x \in E_{X|(Y,Z)=(2,1)} = \{x \in \mathbb{R} \mid (x,2,1) \in E_{\underline{Z}}\} \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{2x}{72P(Y=2,Z=1)}, & \text{if } x \in \{1,2\} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Now, $P(Y = 2, Z = 1) = \sum_{x \in R_{(2,1)}} f(x, 2, 1)$, where $R_{(2,1)} = \{x \in \mathbb{R} \mid (x, 2, 1) \in E_{\underline{Z}}\} = \{1, 2\}$. Hence, $P(Y = 2, Z = 1) = f(1, 2, 1) + f(2, 2, 1) = \frac{1}{12}$. Therefore,

$$f_{X|(Y,Z)}(x|(2,1)) = \begin{cases} \frac{x}{3}, & \text{if } x \in \{1,2\} \\ 0, & \text{otherwise} \end{cases}$$

(2) The conditional p.m.f. of X, given $Y=3$, is

$$f_{(X,Z)|Y}((x,z)|3) = \frac{f(x,3,z)}{P(Y=3)}$$
$$= \begin{cases} \frac{3xz}{72P(Y=3)}, \text{ if } x \in E_{X,Z)|Y=3} = \{(x,z) \in \mathbb{R} \mid (x,3,z) \in E_{\underline{Z}}\} \\ 0, \text{ otherwise} \end{cases}$$
$$= \begin{cases} \frac{3xz}{72P(Y=3)}, \text{ if } (x,z) \in \{1,2\} \times \{1,3\} \\ 0, \text{ otherwise} \end{cases}$$

Now, $P(Y=3) = \sum_{(x,z)\in R_3} f(x,3,z)$, where $R_3 = \{(x,z) \in \mathbb{R} \mid (x,3,z) \in E_{\underline{Z}}\} = \{1,2\} \times \{1,3\}$. Hence, $P(Y=3) = f(1,3,1) + f(1,3,3) + f(2,3,1) + f(2,3,3) = \frac{1}{2}$.
Therefore,

$$f_{(X,Z)|Y}((x,z)|3) = \begin{cases} \frac{xz}{12}, \text{ if } x \in \{1,2\} \times \{1,3\} \\ 0, \text{ otherwise} \end{cases}$$

2. Independent random variables

Definition 5. *The random variables $X_1, X_2, \ldots, X_n$ are said to be independent if for any sub-collection $\{X_{i_1}, X_{i_2}, \ldots, X_{i_k}\}$, $2 \le k \le n$, we have*

$$F_{X_{i_1},\ldots,X_{i_k}}(x_1, x_2, \cdots, x_k) = \prod_{j=1}^{k} F_{X_{i_j}}(x_j), \ \forall \ (x_1, x_2, \cdots, x_k) \in \mathbb{R}^k$$

where $F_{X_{i_1},\ldots,X_{i_k}}$ is the joint c.d.f. of $(X_{i_1}, X_{i_2}, \ldots, X_{i_k})$ and $F_{X_{i_j}}$ is the marginal c.d.f. of X_{i_j}, for $1 \le j \le k$.

Theorem 6. *Let $\underline{X} = (X_1, X_2, \ldots, X_n) : \mathcal{S} \longrightarrow \mathbb{R}^n$ be a $n-$dimensional ($n \ge 2$) random vector with joint c.d.f. $F_{\underline{X}}$. Let F_{X_i} be the marginal c.d.f. of X_i, for $1 \le i \le n$. Then the random variables $X_1, X_2, \ldots, X_n$ are independent if and only if*

$$F_{\underline{X}}(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} F_{X_i}(x_i), \ \forall \ (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n.$$

Theorem 7. *Let $\underline{X} = (X_1, X_2, \ldots, X_n) : \mathcal{S} \longrightarrow \mathbb{R}^n$ be a $n-$dimensional ($n \ge 2$) random vector of either discrete or continuous type. Let $f_{\underline{X}}$ be the joint p.m.f. (or p.d.f.) of $\underline{X}$ and f_{X_i} be the marginal p.m.f. (or p.d.f.) of random variable X_i, for $1 \le i \le n$. Then*

(1) *the random variables $X_1, X_2, \ldots, X_n$ are independent if and only if*

$$f_{\underline{X}}(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i), \ \forall \ (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n.$$

(2) *the random variables $X_1, X_2, \ldots, X_n$ are independent $\Rightarrow E_{\underline{X}} = \prod_{i=1}^{n} E_{X_i}$, where $E_{\underline{X}}$ is the support of random vector $\underline{X}$ and E_{X_i} is the support of random variable X_i, for $1 \le i \le n$.*

Theorem 8. *Let $X_1, X_2, \ldots, X_n$ be the independent random variables.*

(1) *Let $\psi_i : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $\psi_i(A) \in \mathbb{B}_{\mathbb{R}}$, for all $A \in \mathbb{B}_{\mathbb{R}}$, for $i = 1, 2, \cdots, n$. Then the random variables $\psi_1(X_1), \psi_2(X_2), \ldots, \psi_n(X_n)$ are independent.*

(2) *For* $A_i \in \mathbb{B}_{\mathbb{R}}$, $i = 1, 2, \cdots, n$, *we have*

$$P(\{X_i \in A_i, i = 1, 2, \cdots, n\}) = \prod_{i=1}^{n} P(\{X_i \in A_i).$$

Remark 9. $\underline{X} = (X_1, X_2)$ *be a random vector of either discrete or continuous type. Let* $D = \{x_2 \in \mathbb{R} \mid f_{X_1|X_2}(.|x_2)$ *is defined*$\}$. *Then for* $x_2 \in D$, X_1 *and* X_2 *are independent if and only if* $f_{X_1|X_2}(x_1|x_2) = f_{X_1}(x_1)$, *for all* $x_1 \in \mathbb{R}$,

i.e,

X_1 *and* X_2 *are independent if and only if* $\forall\ x_2 \in D$, *the conditional distribution of* X_1, *given* $X_2 = x_2$, *is the same as unconditional distribution of* X_1.

Example 10. *Let* $\underline{Z} = (X, Y, Z)$ *be a random vector with joint p.m.f.*

$$f(x, y, z) = \begin{cases} \frac{xyz}{72}, & \textit{if } (x, y, z) \in \{1, 2\} \times \{1, 2, 3\} \times \{1, 3\} \\ 0, & \textit{otherwise} \end{cases}$$

(1) *Are* X, Y *and* Z *independent random variables?*
(2) *Are* X *and* Z *independent random variables?*

Solution:

(1) The supports of X, Y and Z are

$$E_X = \{x \in \mathbb{R} \mid (x, y, z) \in E_{\underline{Z}} \text{ for some } (y, z) \in \mathbb{R}^2\} = \{1, 2\}$$
$$E_Y = \{y \in \mathbb{R} \mid (x, y, z) \in E_{\underline{Z}} \text{ for some } (x, z) \in \mathbb{R}^2\} = \{1, 2, 3\}$$

and

$$E_Z = \{z \in \mathbb{R} \mid (x, y, z) \in E_{\underline{Z}} \text{ for some } (x, y) \in \mathbb{R}^2\} = \{1, 3\},$$

respectively. For $x \in E_X$, $R_x = \{(y, z) \in \mathbb{R}^2 \mid (x, y, z) \in E_{\underline{Z}}\} = \{1, 2, 3\} \times \{1, 3\}$. So the marginal p.m.f. of X is

$$f_X(x) = \begin{cases} \sum\limits_{(y,z) \in R_x} f(x, y, z), & \text{if } x \in E_X \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{x}{3}, & \text{if } x \in \{1, 2\} \\ 0, & \text{otherwise} \end{cases}$$

Similarly the marginal p.m.f. of Y and Z are

$$f_Y(y) = \begin{cases} \frac{y}{6}, & \text{if } y \in \{1, 2, 3\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_Z(z) = \begin{cases} \frac{z}{4}, & \text{if } y \in \{1, 3\} \\ 0, & \text{otherwise} \end{cases}$$

respectively. Clearly $f(x, y, z) = f_X(x) f_Y(y) f_Z(z)$, for all $(x, y, z) \in \mathbb{R}^3$. Thus X, Y and Z are independent.

(2) Let $\underline{X} = (X, Y)$. The support of $\underline{X}$ is $E_{\underline{X}} = \{(x, z) \in \mathbb{R}^2 \mid (x, y, z) \in E_{\underline{Z}}$ for some $y \in \mathbb{R}\} = \{1, 2\} \times \{1, 3\}$. For $(x, z) \in E_{\underline{X}}$, $R_{(x,z)} = \{x \in \mathbb{R} \mid (x, y, z) \in E_{\underline{Z}}\} = \{1, 2, 3\}$.

So the marginal p.m.f. of $\underline{X}$ is

$$f_{\underline{X}}(x,z) = \begin{cases} \sum\limits_{y \in R_{(x,z)}} f(x,y,z), & \text{if } (x,z) \in E_{\underline{X}} \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{xz}{12}, & \text{if } (x,z) \in \{1,2\} \times \{1,3\} \\ 0, & \text{otherwise} \end{cases}$$

Thus $f_{\underline{X}}(x,z) = f_X(x) f_Z(z)$, for all $(x,z) \in \mathbb{R}^2$. Thus X and Z are independent.

Example 11. *Let $\underline{Z} = (X,Y)$ be a random vector with joint p.d.f.*

$$f_{\underline{Z}}(x,y) = \begin{cases} \frac{1}{x}, & \text{if } 0 < y < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Are X and Y independent?

Solution: By Example 7 of Lecture 14, the marginal p.d.f. of X and Y are

$$f_X(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} -\ln y, & \text{if } 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Clearly, $f_{\underline{Z}}(x,y) \neq f_X(x) f_Y(y)$. Hence, X and Y are not independent.

Alternative solution: The support of $\underline{Z}$ is $E_{\underline{Z}} = \{(x,y) \in \mathbb{R}^2 \mid 0 < y < x < 1\}$, and the support of X and Y are $(0,1)$. Hence, $E_{\underline{Z}} \neq E_X \times E_Y$. Therefore, X and Y are not independent.