Prerequisites

1 Sets

A set can be thought of as a collection of well defined objects. The objects belonging to the set are known as elements of the set. The set and its elements are generally denoted by $A, B, C, \cdots$, and $a, b, c, \ldots$, respectively.

Example 1 *1. The set of first five natural numbers can be written as* $A = \{1, 2, 3, 4, 5\}$ *(Roster form).*

2. A set can also be written as $A = \{x|P(x)\}$*, where* $P(x)$ *is some property. For instance,* $A = \{x|x : x \leq 5, x \in \mathbb{N}\}$*, where* $\mathbb{N}$ *is the set of all natural numbers (Set builder form).*

Definition 1 *1. A set which contains all elements is called universal set. Its is denoted by* U*.*

2. A set which contains no elements is called an empty or null set. It is denoted by ϕ*.*

3. A set A *is said to be a subset of another set* B *if every element of* A *is also an element of* B*.It is denoted as* $A \subseteq B$*. Here,* B *is called a superset of* A*.*

4. A *is a proper subset of* B *if there is at least one element in* B *which does not belong to* A*.*

5. Two sets A *and* B *are said to be equal if* $A \subseteq B$ *and* $B \subseteq A$*. That is, every element of* A *is an element of* B *and vice-versa.*

6. A set A *is called finite if it has finitely many elements.*

1.1 Operations on Sets

1. Complement: The complement of a set A, generally denoted as A^c orA' is given by $A^c = \{x|x \notin A, x \in U\}$.

   Let $\mathcal{I}$ be an index set. Consider a family of sets $\{A_i, i \in \mathcal{I}\}$ indexed by I.

2. Union: The union of $\{A_i, i \in \mathcal{I}\}$ is defined as

$$\cup_{i \in \mathcal{I}} A_i = \{x|x \in A_i \text{ for some } i \in \mathcal{I}\}.$$

   In words, the union $\cup_{i \in \mathcal{I}} A_i$ is a set consisting of those elements which are elements of at least one of the A_i's.

3. Intersection: The intersection of $\{A_i, i \in \mathcal{I}\}$ is defined as

$$\cap_{i\in\mathcal{I}} A_i = \{x | x \in A_i \text{ for every } i \in \mathcal{I}\}.$$

   In words, the intersection $\cap_{i\in\mathcal{I}} A_i$ is a set consisting of those elements which are elements of all A_i's.

4. De Morgan's Law:

$$(\cap_{i\in\mathcal{I}} A_i)^c = \cup_{i\in\mathcal{I}} A_i^c$$
$$(\cup_{i\in\mathcal{I}} A_i)^c = \cap_{i\in\mathcal{I}} A_i^c$$

5. Relative Complement: The relative complement of B in A is defined as $A \setminus B = \{x | x \in A, x \notin B\} = A \cap B^c$. Similarly, the relative complement of A in B is defined as $B \setminus A = \{x | x \in B, x \notin A\} = A^c \cap B$.

6. Symmetric Difference: The symmetric difference of two sets A and B is defined as

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

7. Cartesian product: The Cartesian product of two sets A and B is defined as $A \times B = \{(x, y) : x \in A; y \in B\}$, that is, it is the set of all ordered pairs of elements from the two sets, such that the first component belongs to A and the second to B. For example, $A = \{a, b\}$ and $B = \{1, 2\}$, them $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$. **Is this operation commutative?**

Definition 2 *Power set: The power set of a set A, denoted as $P(A)$ is the set of all subsets of A including the null set and A itself.*

For example, the power set of $A = 1, 2$ is $P(A) = \{\{\phi\}, \{1\}, \{2\}, \{1, 2\}\}$. A power set is an example of a class, which is a collection of sets and is usually denoted by a script letter, $\mathcal{A}$.

2 Functions

A subset of $A \times B$ is said to be a function if it maps every element of set A to a unique element in set B. In other words, $\forall\ x \in A$; $\exists\ y \in B$ and only one such element, such that, $f(x) = y$. Then y is called the image of x and x, the pre-image of y under f. The set A is called the domain of the function and B, the co-domain. $R = \{y : \exists\ x \in A, s.t. f(x) = y\}$ is called as the range of the function f.

1. Injective (one-one) function: A function $f : A \to B$ is said to be an injective (one-to-one) function, if every element in the range R has a unique pre-image in A.

2. Surjective (onto) function: A function $f : A \to B$ is said to be a surjective (onto) function, if $R = B$, i.e, $\forall\ y \in B;\ \exists\ x \in A; s.t. f(x) = y$.

3. Bijective function: A function $f : A \to B$ is a bijective function if it is both injective and surjective.

   Hence, in a bijective mapping, every element in the co-domain has a pre-image and the pre-images are unique. Thus, we can define an inverse function, such that, $f^{-1} : B \to A$ such that $f^{-1}(y) = x$, if $f(x) = y$.

2.1 Cardinality and Countability

The cardinality of a set is the number of elements in that set, if the set is finite. It is denoted by $|.|$.

(a) Two sets A and B are equicardinal (denoted as $|A| = |B|$) if there exists a bijective function from A to B.

(b) B has cardinality greater than or equal to that of A (denoted as $|B| \geq |A|$) if there exists an injective function from A to B.

(c) B has cardinality strictly greater than that of A (denoted as $|B| > |A|$) if there is an injective function, but no bijective function, from A to B.

(d) A set E is said to be countably infinite if E and $\mathbb{N}$ are equicardinal.

Example 2 *(a) The set of all integers $\mathbb{Z}$ is countably infinite.*

(b) The set of all rationals $\mathbb{Q}$ is countably infinite.

Remark: For two finite sets A and B; we know that if A is a proper subset of B; then B has cardinality strictly greater than that of A. But, this is not true for infinite sets. Indeed, $\mathbb{N}$ is a proper subset of $\mathbb{Q}$; but $\mathbb{N}$ and $\mathbb{Q}$ are equicardinal. (Prove yourself!)

Definition 3 *(a) A set is said to be countable if it is either finite or countably infinite.*

(b) A set A is said to be uncountable if it has cardinality strictly greater than the cardinality of $\mathbb{N}$, that is, there does not exists a bijection from $\mathbb{N}$ to A.

Theorem 4 *Let $\mathcal{I}$ be a countable index set, and let E_i be countable for each $i \in \mathcal{I}$. Then, $\cup_{i \in \mathcal{I}} E_i$ is countable. That is, a countable union of countable sets is countable.*

Example 3 *The sets $\mathbb{R}$, $[0,1]$ and $\mathbb{R} \setminus \mathbb{Q}$ are uncountable.*

3 Real Analysis

3.1 Order Axioms

(a) Law of trichotomy: If $a, b \in \mathbb{R}$, then $a = b$ or $a > b$ or $a < b$.

(b) Transitivity: Let $a, b, c \in \mathbb{R}$. If $a > b$ and $b > c$, then $a > c$.

(c) Ordering and addition operator: Let $a, b, c \in \mathbb{R}$. $a > b \Rightarrow a + c > b + c$.

(d) Ordering and product operator: Let $a, b, c \in \mathbb{R}$. $a > b \Rightarrow ac > bc$, if $c > 0$.

3.2 Boundedness

A subset S of $\mathbb{R}$ is bounded above if $\exists\ M \in \mathbb{R}$ such that $x \leq M, \forall\ x \in S$. Here, M is called an upper bound of S. Similarly S is bounded below if $\exists$ $m \in \mathbb{R}$ such that $x \geq m, \forall\ x \in S$. Here, m is called a lower bound of S. A set is bounded if it is both bounded above and below. Any real number greater than M and lesser than m are also upper and lower bounds of S respectively.

Definition 5 *Supremum: The supremum of S is the least upper bound of the set S. More precisely, K is the supremum of S if*

- *K is an upper bound of S, i.e. , $x \leq K$, $\forall\ x \in S$.*
- *There exists no number less than K which is an upper bound of S, i.e. for any $\delta > 0$, $\exists\ z \in S$ such that $z > K - \delta$.*

Similarly, the infimum of S is defined as the greatest lower bound of the set S.

Remark 6 *The supremum and infimum need not be elements of the set. For instance, 1 is the supremum of the set $(0,1)$, but is not an element of the set. Also, if the supremum and infimum are an element of the set itself, these are called the maximum and minimum of that set, respectively.*

3.3 Completeness property

The completeness axiom or the least upper bound property is one of the fundamental properties of the real field $\mathbb{R}$.
Completeness Axiom: Any non empty subset A of $\mathbb{R}$ which is bounded above(below) has a supremum(infimum) in $\mathbb{R}$.
In other words, the Completeness Axiom guarantees that, for any nonempty set of $\mathbb{R}$ that is bounded above(below), the supremum(infimum) exists.

3.4 Sequences

Definition 7 *A (real) sequence is a function from $\mathbb{N}$ to $\mathbb{R}$. A sequence (x_n) of real numbers is said to converge to $x \in \mathbb{R}$ if for every $\epsilon > 0$, exists $n_0 \in \mathbb{N}$ such that $|x_n - x| < \epsilon \ \forall \ n \geq n_0$. It is denoted as $x_n \to x$.*

Theorem 8 *Let (x_n) be a monotonically increasing sequence, i.e., $x_n \leq x_{n+1} \ \forall \ n$ and bounded above. Then (x_n) converges to a real number (its supremum).*

Theorem 9 *A convergent sequence is bounded but the converse need not be true.*

Theorem 10 *Sandwich Theorem: Let $(x_n), (y_n)$ and (z_n) be three sequences such that $x_n \leq z_n \leq y_n \ \forall \ n$, and if $x_n \to x$ and $y_n \to x$, then $z_n \to x$.*

Definition 11 *Cauchy Sequences: A sequence (x_n) is called a Cauchy sequence if $\forall \ \epsilon > 0$, $\exists \ n_0 \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon \ \forall \ n, m \geq n_0$.*

Theorem 12 *A real sequence is convergent if and only if it is a Cauchy sequence.*

Definition 13 *Subsequence: Let (x_n) be a sequence and let (n_k) be any sequence of positive integers such that $n_1 < n_2 < n_3 < \ldots$. The sequence (x_{n_k}) is called a subsequence. In other words, a subsequence of a sequence is an infinite ordered subset of that sequence.*

Theorem 14 *Bolzano-Weistrass Theorem: Every bounded sequence has a convergent subsequence.*

Theorem 15 *A sequence (x_n) is convergent if and only if (x_n) is bounded and every convergent subsequence of (x_n) converges to the same limit.*