Indian Institute of Information Technology Allahabad
B. Tech. 3rd Semester
Probability and Statistics (PAS)
C3 Review Test

Date: November 23, 2023(3:00 PM - 5:00 PM) Total Marks: 40

Important Instructions:

1. Attempt all questions. There is no credit for a solution if the appropriate work is not shown, even if the answer is correct. All the notations are standard and same as used in the lecture notes.
2. Writing on question paper is not allowed.
3. Attempt all the parts of a question at the same place. Parts done separately will not be graded.
4. Number the pages of your answer booklet. On the back of the front page of your answer booklet, make a table (as shown below) to indicate the page number in which respective questions have been answered. If you did not attempt a particular question, write down NA.

Question No.	1	2	3	4	5	6
Page No.						

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1. A and B play until one has 2 more points than the other. Assuming that each point is independently won by A and B with probability p and $1-p$, respectively. What is the probability that they will play a total of $2n$ points? What is the probability that A will win? [6]

   Solution: Let trial 1 consist of the first two points; trial 2 the next two points, and so on. Thus, for $2n$ points the game runs for n trials.

   The probability that each player score one point in a trial is $2p(1-p)$. [1]

   Now a total of $2n$ points are played if the first $(n-1)$ trials all result in each player getting one of the points in that trial and the n-th trial results in one of the players winning both points.

   The probability that A wins on trial n is $(2p(1-p))^{n-1}p^2$. [1]

   The probability that B wins on trial n is $(2p(1-p))^{n-1}(1-p)^2$. [1]

   By independence, we obtain $P\{$ the players will play a total of $2n$ points $\} = (2p(1-p))^{n-1}(p^2+(1-p)^2)$. [1]

   So $P\{A \text{ wins }\} = p^2\sum_{n=1}^{\infty}(2p(1-p))^{n-1} = \frac{p^2}{1-2p(1-p)}$ [2]

2. If the number of typographical errors per page type by a certain typist follows a Poisson distribution with mean value λ, find the probability that the total number of errors in 10 randomly selected pages is 10. [4]

   Solution: Let X_i be the number of typographical errors on a randomly selected page.

   Given that $X_i \sim P(\lambda)$.

   Let Y be the total number of errors in 10 randomly selected pages. Then $Y = \sum_{i=1}^{10} X_i$. Since X_i are independent, $Y \sim P(10\lambda)$. [1+1]

   $P(Y=10) = \frac{e^{-10\lambda}(10\lambda)^{10}}{10!}$ [1+1]

3. The joint probability mass function of two discrete random variables X and Y is given by [10]

$$f(x,y) = \begin{cases} c(2x+y) & \text{if } (x,y) \in \{0,1,2\} \times \{0,1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the value of c.

   (b) Evaluate $E(X|Y=3)$ and $Var(X|Y=3)$.

Solution: Since $\sum_{x=0}^{2}\sum_{y=0}^{3} f_{X,Y}(x,y) = 1$, we have $\sum_{x=0}^{2}\sum_{y=0}^{3} c(2x+y) = 1 \Rightarrow c = \frac{1}{42}$. [1+1]

$f_Y(y=3) = \sum_{x=0}^{2} \frac{1}{42}(2x+3) = \frac{15}{42}$. [1 + 1]

The conditional PMF of $X|Y=3$ is given by

$f_{X|Y}(x|y=3) = \frac{f_{X,Y}(x,3)}{f_Y(3)} = \frac{2x+3}{15}, x = \{0,1,2\}$, 0, otherwise. [1+1]

$E(X|Y=3) = \sum_{x=0}^{2} x f_{X|Y}(x|y=3) = \frac{19}{15}$. [1]

$E(X^2|Y=3) = \sum_{x=0}^{2} x^2 f_{X|Y}(x|y=3) = \frac{33}{15}$. [1]

Hence, $V(X|Y=3) = E(X^2|Y=3) - (E(X|Y))^2 = \frac{134}{225}$. [1+1]

4. Let (X,Y) be a two-dimensional random vector having the joint probability density function

$$f(x,y) = \begin{cases} 4xye^{-(x^2+y^2)}, & x>0, y>0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the probability density function of $\sqrt{X^2+Y^2}$ using the transformation of variables technique. [8]

Solution: Let $u = \sqrt{x^2+y^2}$ and $v = x$. This implies $x = v$ and $y = \sqrt{u^2-v^2}$. [1]

The Jacobian of transformation is given by

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{-u}{\sqrt{u^2-v^2}}.[1]$$

The joint PDF of $U = \sqrt{X^2+Y^2}$ and $V = X$ is given by $g(u,v) = f(x,y)|J|$. [1]

$$g(u,v) = \begin{cases} 4uve^{-u^2}, & u>0,\ 0<v<u \\ 0, & \text{otherwise.} \end{cases} \qquad [1+1]$$

The marginal PDF of U is given by $f(u) = \int_0^u g(u,v)dv$. [1]

$$f(u) = \begin{cases} 2u^3e^{-u^2}, & u>0 \\ 0, & \text{otherwise.} \end{cases} \qquad [1+1]$$

5. Let $X_1, \ldots, X_{100}$ be independent and identically distributed (iid) RVs with mean 75 and variance 225. Use Chebychev's inequality to calculate the probability that the sample mean will not differ from the population mean by more than 6. Then use the central limit theorem (CLT) to calculate the same probability, and compare your results. [8]

Solution: Let $E(\frac{X_1+...+X_{100}}{100}) = 75$ and $Var(\frac{X_1+...+X_{100}}{100}) = \frac{225}{100} = \frac{9}{4}$. [2]

By Chebychev's inequality, $P\left(|\frac{X_1+...+X_{100}}{100}) - 75| \geq 6\right) \leq \frac{1}{16}$. [1]

Hence, $\frac{15}{16} \leq P\left(|\frac{X_1+...+X_{100}}{100}) - 75| < 6\right) \leq 1$. [1]

By the central limit theorem, $\frac{\frac{X_1+...+X_{100}}{100})-75}{\frac{3}{2}} \sim Z = N(0,1)$. [1]

So, $P\left(|\frac{X_1+...+X_{100}}{100}) - 75| < 6\right) = P\left(|\frac{\frac{X_1+...+X_{100}}{100})-75}{\frac{3}{2}}| < 4\right) = P(|Z| < 4) = \Phi(4) - \Phi(-4) = 2\Phi(4) - 1 = 1$. [1+1+1]

6. Let $\{X_n\}$ be a sequence of RVs defined by $P(X_n = 0) = 1 - \frac{1}{n^r}$ and $P(X_n = n) = \frac{1}{n^r}, \quad r > 0, \quad n = 1, 2, \ldots$. Find a real number a such that $X_n \xrightarrow{p} a$. [4]

Solution: Let $\epsilon > 0$. Then

$$P(|X_n| \geq \epsilon) = \begin{cases} P(X_n = n) = \frac{1}{n^r}, & \text{if } \epsilon \leq n \\ 0, & \text{if } \epsilon > n. \end{cases} \qquad [2]$$

Thus $\lim_{n\to\infty} P(|X_n| \geq \epsilon) = 0$, that is, $X_n \xrightarrow{p} 0$. [2]

OR

$E(X_n) = E(|X_n|) = \frac{1}{n^{r-1}}$. [1]

Let $\epsilon > 0$. By Markov's Inequality, $P(|X_n| \geq \epsilon) \leq \frac{E(|X_n|)}{\epsilon} = \frac{1}{\epsilon n^{r-1}}$. [1]

Assume $r > 1$. Thus $\lim_{n\to\infty} P(|X_n| \geq \epsilon) = 0$, that is, $X_n \xrightarrow{p} 0$. [2]

OR

Assume $r > 2$. Then $E(X_n) = \frac{1}{n^{r-1}}$ and $E(X_n^2) = \frac{1}{n^{r-2}}$ if $r > 2$. Also, $Var(X_n) = E(X_n^2) - (E(X_n))^2 = \frac{1}{n^{r-2}} - \frac{1}{n^{2(r-1)}}$. [1+1]

Since $\lim_{n\to\infty} E(X_n) = 0$ and $\lim_{n\to\infty} Var(X_n) = 0$. So, $X_n \xrightarrow{p} 0$. [1]

This method is not applicable for $r = 1, 2$. [4-1]