Q. 1

(½) (½) 1

$$P(Y \le y) = P(\sin X \le y), \quad 0 < y < 1$$

$$= P(\{X \le \sin^{-1} y\} \cup \{\pi - \sin^{-1} y \le X \le \pi\})$$

(1) (1)

$$= \frac{2}{\pi^2}\left(\int_0^{\sin^{-1} y} x\,dx + \int_{\pi - \sin^{-1} y}^{\pi} x\,dx\right)$$

$$= \frac{2}{\pi^2}\left[\frac{x^2}{2}\Bigg|_0^{\sin^{-1} y} + \frac{x^2}{2}\Bigg|_{\pi - \sin^{-1} y}^{\pi}\right]$$

$$= \frac{2}{\pi^2}\left[(\sin^{-1} y)^2 + \pi^2 - (\pi - \sin^{-1} y)^2\right]$$

$$= \frac{1}{\pi^2}\left[(\sin^{-1} y)^2 + \pi^2 - (\pi^2 + (\sin^{-1} y)^2 - 2\pi \sin^{-1} y)\right]$$

$$= \frac{1}{\pi^2}\left[2\pi \sin^{-1} y\right]$$

$$= \frac{2\sin^{-1} y}{\pi}$$

2

$$f_Y(y) = \frac{2}{\pi} \frac{1}{\sqrt{1-y^2}} \qquad 0 < y < 1$$

$$0 \qquad \text{o.w}$$

2

(1)

Question 2.

$$F(x,y) = \begin{cases} 1 & y \geq e^{-x} > 0 \\ 0 & \text{o.w.} \end{cases}$$

(i) $\lim_{x \to -\infty} F(x,y) = 0$ $\Bigg\}$ $x \to -\infty,\ e^{-x} \to \infty$, $y < e^{-x}$, $F(x,y) = 0$ ½ ✓ (1)

$\lim_{y \to -\infty} F(x,y) = 0$

(ii) $\lim_{x, y \to (\infty,\infty)} F(x,y) = 1.$ (1)

$e^{-\infty} \to 0$, $y \to \infty$

$\Rightarrow$ $y \geq e^{-x} \Rightarrow F(x,y) = 1$

(iii) Fix $x = x_0$

$$F(x_0,y) = \begin{cases} 1 & y \geq e^{-x_0} \\ 0 & \text{o.w.} \end{cases}$$

y_1, y_2 s.t $y_1 < y_2$

If $y_1 \leq e^{-x_0} < y_2$

$\Rightarrow$ $F(y_1) = 0$ $F(y_2) = 1$

If $e^{-x_0} \leq y_1 < y_2$

$F(y_1) = F(y_2) = 1$

If $y_1 < y_2 < e^{-x_0}$ $F(y_1) = F(y_2) = 0$

Nondec

| $y = e^{-x_0}$
| C.H.L
| Right cty
| $\lim_{h \to 0} F(y+h)$

(1½)

For $y = y_0$

$$F(x) = F(x, y_0) = \begin{cases} 1 & y_0 \geq e^{-x_0} \\ 0 & o.\omega \end{cases}$$

$$x_1 < x_2 \Rightarrow e^{-x_2} < e^{-x_1} < y_0$$

$$F(x_1) = F(x_2) = 0$$

$$y_0 < e^{-x_2} < e^{-x_1}$$

$$F(x_1) = F(x_2) = 1$$

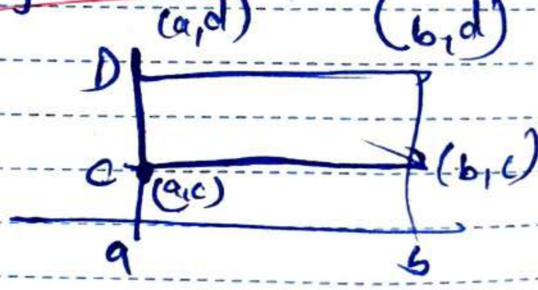
1/2

$$e^{-x_2} < y_0 < e^{-x_1}$$

$$F(x_2) = 1 \quad , \quad F(x_1) = 0$$

$\Rightarrow$ Non-decreasing

$(a < b)$ $(c < d)$



$\cancel{F(a,d) + F(b,c) - F(b}$

$$F(a,c) + F(b,d) - F(a,d) - F(b,c) > 0$$

1

Choose the points.

$(0,1)$ $(0,1)$

$$F(0,0) + F(1,1) - F(0,1) - F(1,0)$$

$$0 + 1 - 1$$

1

$1 \geq e^{-1}$

$1 \geq e^{0}$

$1 \geq e^{-1/2}$

$a = 0,\ b = 1,\ c = \frac{1}{2},\ d = 1$

$f(1,1) + f(0,\tfrac{1}{2}) - f(0,1) - f(1,\tfrac{1}{2})$

1

$1 + 0 - 1 - 1 = -1 < 0$

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(8)

$$F(y) = \begin{cases} 1 & y \geq e^{-x_0} \\ 0 & \text{o.w.} \end{cases}$$

$$\lim_{h \to 0^+} F(y+h) = \lim_{h \to 0^+} 1 \qquad y+h > y \geq e^{-x_0}$$

(1/2)

$$= F(y)$$

$$F(x+h) = \lim_{h \to 0^+}$$

(1/2)

Right cts.

$x < x+h$

$e^{-(x+h)} < e^{-x} < 0$

(4) $E(X_i) = 2 \text{ min} = \mu$ — 1

$Var(X_i) = 1 \text{ min} = \sigma^2$ — 1

$Y = X_1 + X_2 + \cdots + X_{50}$

$$P(90 < Y < 110) = P\left(\frac{90 - n\mu}{\sqrt{n}\,\sigma} < \frac{Y - n\mu}{\sqrt{n}\,\sigma} < \frac{110 - n\mu}{\sqrt{n}\,\sigma}\right)$$ — 1

$$= P\left(\frac{90-100}{\sqrt{50}} < \frac{Y - n\mu}{\sqrt{n}\,\sigma} < \frac{110-100}{\sqrt{50}}\right)$$

$$= P\left(-\sqrt{2} < \frac{Y - n\mu}{\sqrt{n}\,\sigma} < \sqrt{2}\right)$$

$$= \phi(\sqrt{2}) - \phi(-\sqrt{2})$$

$$= 2\phi(\sqrt{2}) - 1$$

$$= 2 \times 0.92073 - 1$$

$$= 0.84146 \sim 0.84$$ — 1

(3) $$f_{X,Y}(x,y) = \begin{cases} \frac{c}{x!\,y!\,(2-x-y)!}\left(\frac{1}{6}\right)^x\left(\frac{2}{6}\right)^y\left(\frac{3}{6}\right)^{2-x-y}, & 0 \leq x+y \leq 2 \\ 0 & , \text{o.w.} \end{cases}$$

(a) $$\sum_{x=0}^{2}\sum_{y=0}^{2-x} f_{X,Y}(x,y) = 1$$

$$\sum_{x=0}^{2}\sum_{y=0}^{2-x} \frac{c}{x!\,y!\,(2-x-y)!}\left(\frac{1}{6}\right)^x\left(\frac{2}{6}\right)^y\left(\frac{3}{6}\right)^{2-x-y} = 1$$

$x=0,\ y=0$

$$f_{X,Y}(0,0) = \frac{c}{2}\,\frac{1}{2^2} = \frac{c}{8}$$

$x=0,\ y=1$

$$f_{X,Y}(0,1) = \frac{c}{1} \times \frac{1}{3} \times \frac{1}{2} = \frac{c}{6}$$

$x=0,\ y=2$

$$f_{X,Y}(0,2) = \frac{c}{2}\left(\frac{1}{3}\right)^2 = \frac{c}{18}$$

$x=1, y=0$

$f_{X,Y}(1,0) = \frac{c}{1} \times \frac{1}{6} \times \frac{1}{2} = \frac{c}{12}$

$x=1, y=1$

$f_{X,Y}(1,1) = \frac{c}{1} \times \frac{1}{6} \times \frac{1}{3} = \frac{c}{18}$

$x=2, y=0$

$f_{X,Y}(2,0) = \frac{c}{2} \times \frac{1}{36} = \frac{c}{72}$

$\Rightarrow \frac{c}{8} + \frac{c}{6} + \frac{c}{18} + \frac{c}{12} + \frac{c}{18} + \frac{c}{72} = 1$

$\Rightarrow \frac{9c + 12c + 4c + 6c + 4c + c}{72} = 1$

$\Rightarrow \frac{36c}{72} = 1 \Rightarrow \boxed{c = 2}$ — 3

(b) $f_X(x) = \sum_{y=0}^{2-x} f_{XY}(x,y)$

$f_X(0) = f_{X,Y}(0,0) + f_{X,Y}(0,1) + f_{X,Y}(0,2)$

$= \frac{1}{4} + \frac{1}{3} + \frac{1}{9} = \frac{25}{36}$

$f_X(1) = f_{X,Y}(1,0) + f_{X,Y}(1,1) = \frac{1}{6} + \frac{1}{9} = \frac{5}{18}$

$f_X(2) = f_{X,Y}(2,0) = \frac{1}{36}$

$$f_X(x) = \begin{cases} \frac{25}{36}, & x=0 \\ \frac{5}{18}, & x=1 \\ \frac{1}{36}, & x=2 \\ 0, & \text{o.w.} \end{cases}$$ — 3

$$f_Y(y) = \sum_{x=0}^{2-y} f_{X,Y}(x,y)$$

$$f_Y(0) = f_{X,Y}(0,0) + f_{X,Y}(1,0) + f_{X,Y}(2,0)$$
$$= \frac{1}{4} + \frac{1}{6} + \frac{1}{36} = \frac{16}{36} = \frac{4}{9}$$

$$f_Y(1) = f_{X,Y}(0,1) + f_{X,Y}(1,1)$$
$$= \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$$

$$f_Y(2) = f_{X,Y}(0,2)$$
$$= \frac{1}{9}$$

$$f_Y(y) = \begin{cases} \frac{4}{9} & , y = 0,1 \\ \frac{1}{9} & , y = 2 \\ 0 & , \text{o.w.} \end{cases}$$ — 2

(c) $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

$$f_{X,Y}(0,0) = \frac{1}{4}$$

$$f_X(0) = \frac{25}{36} \qquad f_Y(y) = \frac{4}{9}$$

$$f_X(0) \cdot f_Y(0) = \frac{25}{36} \times \frac{4}{9} = \frac{25}{81} \neq f_{X,Y}(0,0)$$

$\Rightarrow$ Not independent — 1

Question 5:-

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

Likelihood :- $L(\lambda | \underline{x}) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$

$$\frac{\partial}{\partial \lambda} \log L(\lambda | \underline{x}) = n \log \lambda - \lambda \sum x_i \quad \text{— } 1$$

$$\frac{\partial}{\partial \lambda} \log L = \frac{n}{\lambda} - \sum x_i \doteq 0 \Rightarrow \lambda = \frac{1}{\bar{x}}$$

$$\frac{\partial^2}{\partial \lambda^2} \log L \Big|_{\lambda = 1/\bar{x}} = -\frac{n}{\lambda^2} < 0 \quad \text{— } 1$$

MLE of $\lambda = \frac{1}{\bar{x}}$.

$\Rightarrow$ By invariance property, MLE of $\frac{1}{\lambda} = \bar{x}$. — 1

To check whether $\bar{X}$ is unbiased & consistent for $\frac{1}{\lambda}$:-

$$E(\bar{X}) = E\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(X_i)$$

$$= \frac{1}{n} \cdot n \cdot \frac{1}{\lambda} = \frac{1}{\lambda}. \quad \text{— } 1$$

Hence, $\bar{X}$ is unbiased for $\frac{1}{\lambda}$.

To check for consistancy:-

If $\{T_n\}$ is consistent for $g(\theta)$ →

- $\lim_{n \to \infty} E(T_n) = g(\theta)$
- $\lim_{n \to \infty} Var(T_n) = 0$

— 1

$T_n = \bar{X}$.

Since $E(\bar{X}) = \frac{1}{\lambda} \Rightarrow \lim_{n\to\infty} E(\bar{X}) = \frac{1}{\lambda}$.

To check for $Var(\bar{X})$.

$$\Rightarrow Var(\bar{X}) = \frac{1}{n} Var(X)$$

$$Var(X) = \frac{1}{\lambda^2} \Rightarrow Var(\bar{X}) = \frac{1}{n\lambda^2}$$

$$\Rightarrow \lim_{n\to\infty} Var(\bar{X}) = 0.$$ 1

Hence, $\bar{X}$ is unbiased & consistent for $\frac{1}{\lambda}$.

6]

X: 18 13 18 15 10 12 8 4 7 3

Y: 23 20 18 16 14 11 10 7 6 4

Normal eqns:-

$$\sum_{i=1}^{n} x_i = na + b\sum_{i=1}^{n} y_i \quad , n = 10.$$

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} y_i + b\sum_{i=1}^{n} y_i^2$$ 1

X	Y	Y^2	XY
18	23	529	414
13	20	400	260
18	18	324	324
15	16	256	240
10	14	196	140
12	11	121	132
8	10	100	80
4	7	49	28
7	6	36	42
3	4	16	12
108	129	2027	1672

On substituting the values:→

$108 = 10a + 129b$. 2

$1672 = 129a + 2027b$.

Solving for a & b →

$a = 0.89, \quad b = 0.77$ 1+1

Regression line →

$$\boxed{X = 0.89 + 0.77Y}$$ 1