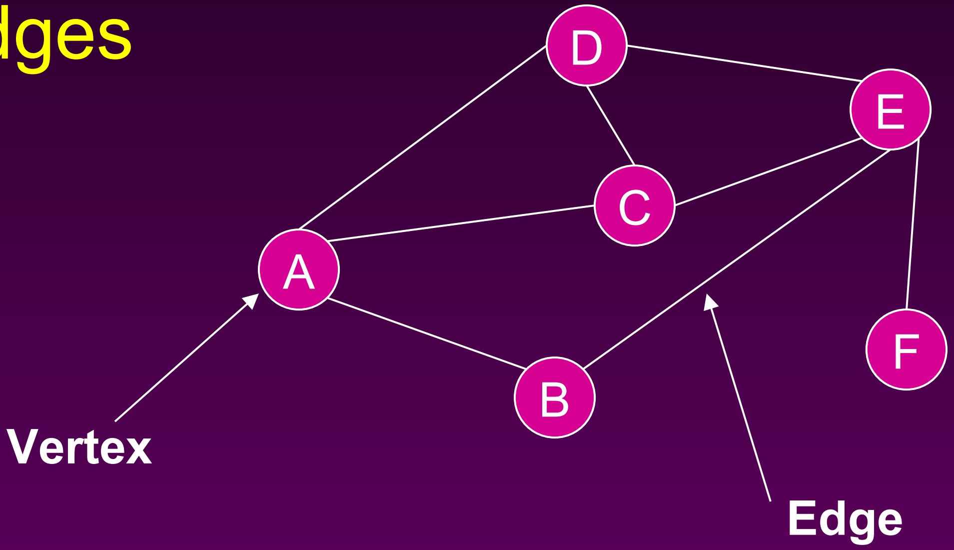
# Graph & BFS

# Graphs

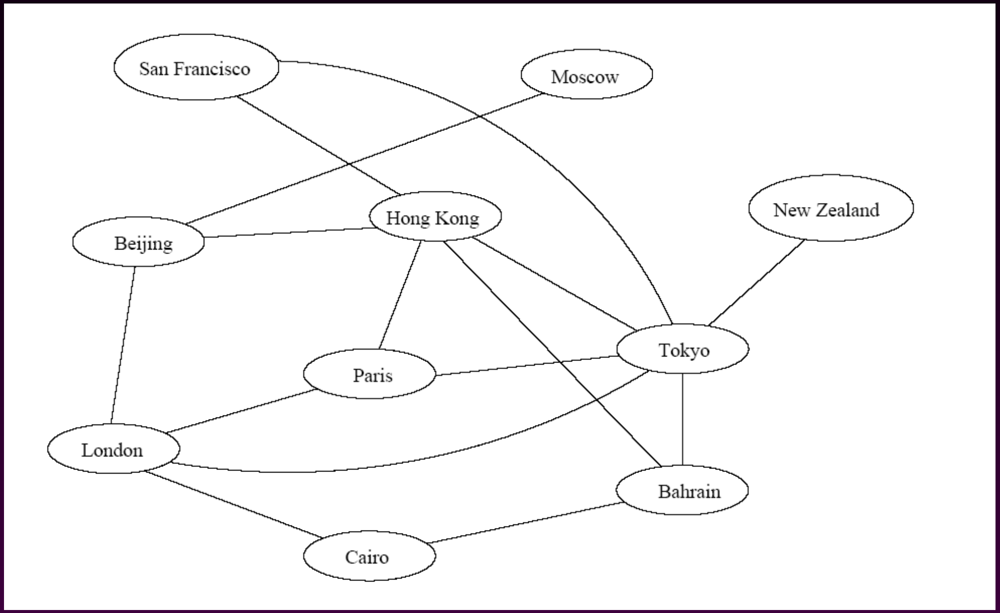
- Extremely useful tool in modeling problems
- Consist of:
  - Vertices
  - Edges



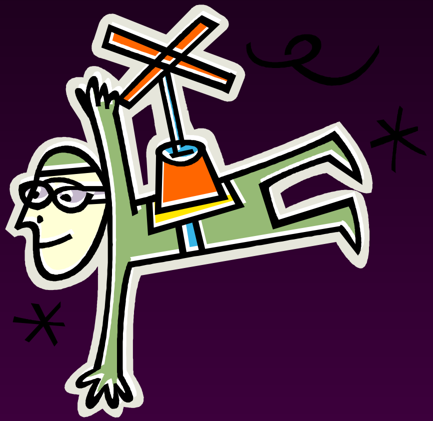
**Vertices** can be considered “sites” or locations.

**Edges** represent connections.

# Application 1

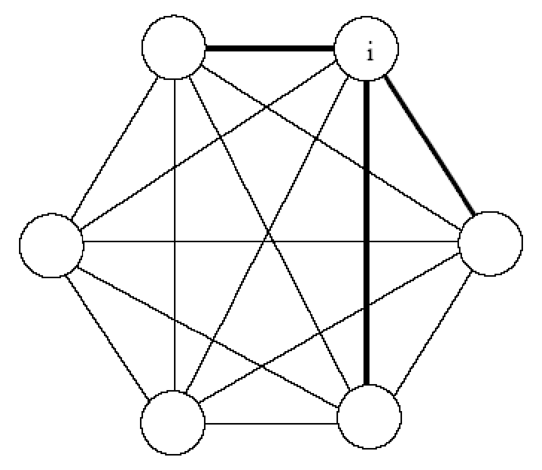


Air flight system

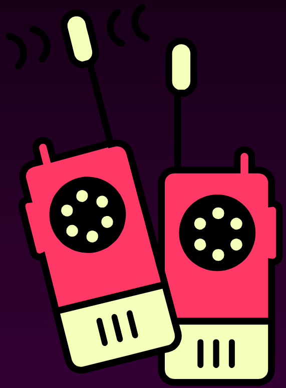


- Each vertex represents a city
- Each edge represents a direct flight between two cities
- A query on direct flights = a query on whether an edge exists
- A query on how to get to a location = does a path exist from A to B
- We can even associate costs to edges (weighted graphs), then ask "what is the cheapest path from A to B"

# Application 2



Wireless communication

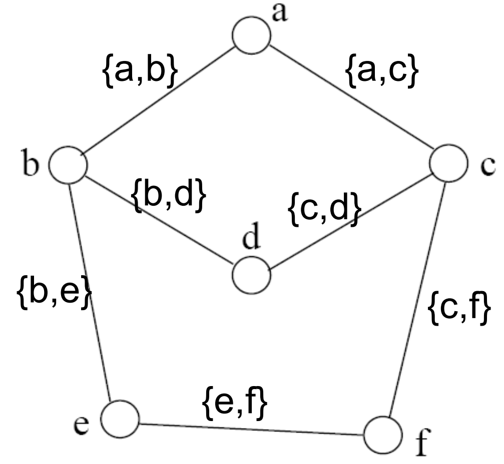


-  Represented by a weighted complete graph (every two vertices are connected by an edge)
-  Each edge represents the Euclidean distance *dij* between two stations
-  Each station uses a certain power i to transmit messages. Given this power i, only a few nodes can be reached (bold edges). A station reachable by i then uses its own power to relay the message to other stations not reachable by i.
-  A typical wireless communication problem is: how to broadcast between *all* stations such that they are all connected and the power consumption is minimized.

- Graph, also called network (particularly when a weight is assgned to an edge)
- A tree is a connected graph  with no loops.
- Graph algorithms might be very difficult!
  - four color problem for planar graph!
- 171 only handles the simplest ones
  - Traversal, BFS, DFS
  - ((Minimum) spanning tree)
  - Shortest paths from the source
  - Connected components, topological sort

# Definition

- A graph G=(V, E) consists a set of vertices, V, and a set of edges, E.
- Each edge is a pair of *(v, w)*, where v, w belongs to V
- If the pair is unordered, the graph is undirected; otherwise it is directed



$$V = \{a, b, c, d, e, f\}$$

$$E = \{\{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}, \{b, e\}, \{c, f\}, \{e, f\}\}$$

**An undirected graph**

# Terminology

1. If $v_1$ and $v_2$ are connected, they are said to be adjacent vertices
   - **$v_1$** and **$v_2$** are endpoints of the edge {**$v_1$**, **$v_2$**}

2. If an edge *e* is connected to *v*, then *v* is said to be incident on *e*.  Also, the edge *e* is said to be incident on *v*.

If we are talking about directed graphs, where edges have direction.  This means that $\{v_1,v_2\} \neq \{v_2,v_1\}$ .  Directed graphs are drawn with arrows (called arcs) between edges.

This means {A,B} only, not {B,A}

# Graph Representation

- Two popular computer representations of a graph. Both represent the vertex set and the edge set, but in different ways.

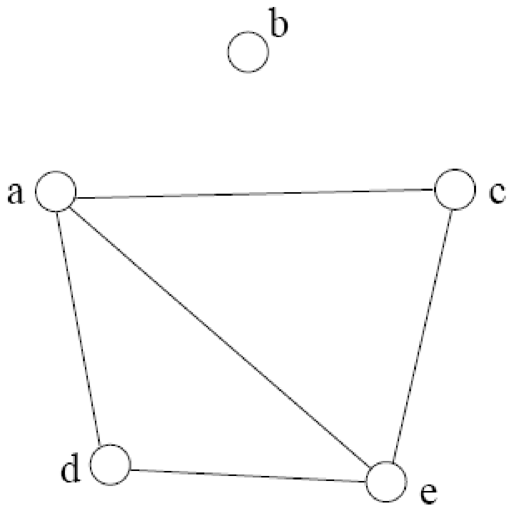
1. Adjacency Matrix

   Use a 2D matrix to represent the graph

2. Adjacency List

   Use a 1D array of linked lists

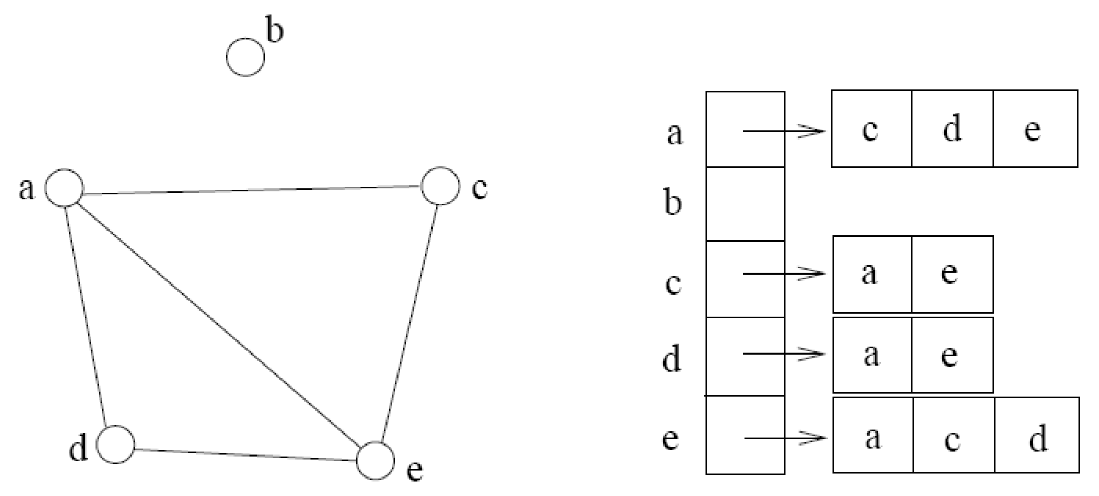
# Adjacency Matrix



$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 0 & 1 & 1 & 1 \\ b & 0 & 0 & 0 & 0 & 0 \\ c & 1 & 0 & 0 & 0 & 1 \\ d & 1 & 0 & 0 & 0 & 1 \\ e & 1 & 0 & 1 & 1 & 0 \end{array}$$

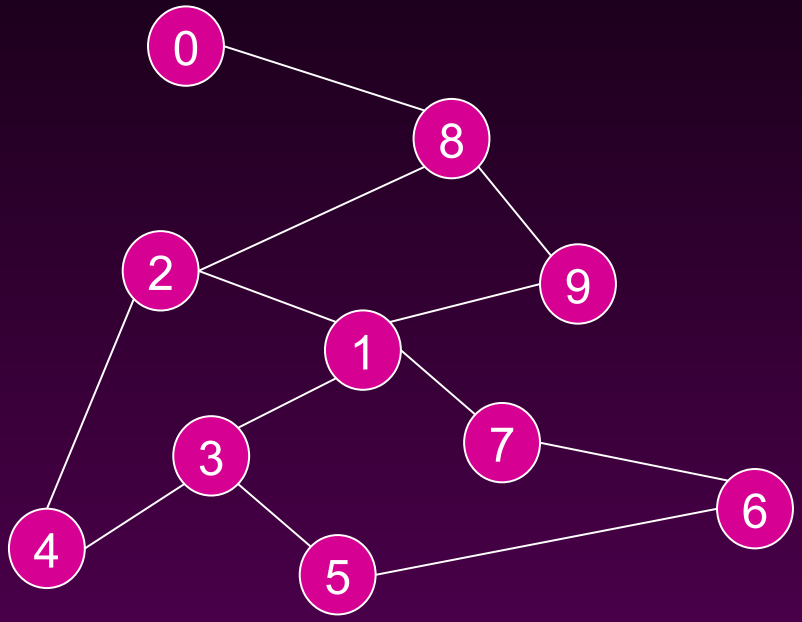
- 2D array A[0..n-1, 0..n-1], where ***n*** is the number of vertices in the graph
- Each row and column is indexed by the vertex id
  - e,g a=0, b=1, c=2, d=3, e=4
- A[i][j]=1 if there is an edge connecting vertices *i* and *j*; otherwise, A[i][j]=0
- The storage requirement is $\Theta(n^2)$. It is not efficient if the graph has few edges. An adjacency matrix is an appropriate representation if the graph is dense: $|E|=\Theta(|V|^2)$
- We can detect in O(1) time whether two vertices are connected.

# Adjacency List



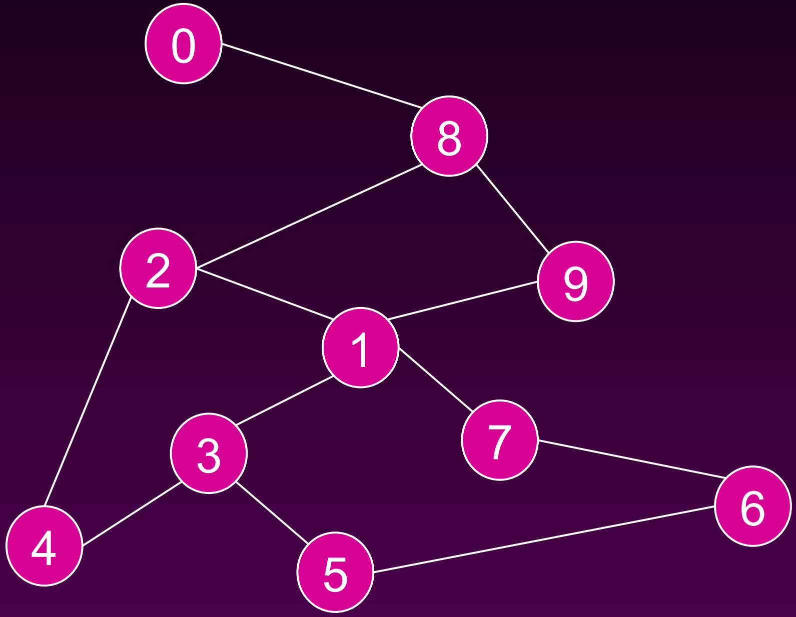
- If the graph is not dense, in other words, sparse, a better solution is an adjacency list
- The adjacency list is an array A[0..n-1] of lists, where n is the number of vertices in the graph.
- Each array entry is indexed by the vertex id
- Each list *A[i]* stores the ids of the vertices adjacent to vertex *i*

# Adjacency Matrix Example



| | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
|---|---|---|---|---|---|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| **1** | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| **2** | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| **3** | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| **4** | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| **5** | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| **6** | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| **7** | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| **8** | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| **9** | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

# Adjacency List Example



| | |
|---|---|
| **0** | 8 |
| **1** | 2 3 7 9 |
| **2** | 1 4 8 |
| **3** | 1 4 5 |
| **4** | 2 3 |
| **5** | 3 6 |
| **6** | 5 7 |
| **7** | 1 6 |
| **8** | 0 2 9 |
| **9** | 1 8 |

# Storage of Adjacency List

- The array takes up $\Theta(n)$ space
- Define degree of $v$, $\deg(v)$, to be the number of edges incident to $v$. Then, the total space to store the graph is proportional to:

$$\sum_{\text{vertex } v} \deg(v)$$

- An edge $e=\{u,v\}$ of the graph contributes a count of 1 to $\deg(u)$ and contributes a count 1 to $\deg(v)$
- Therefore, $\Sigma_{\text{vertex } v}\deg(v) = 2m$, where $m$ is the total number of edges
- In all, the adjacency list takes up $\Theta(n+m)$ space
  - If $m = O(n^2)$ (i.e. dense graphs), both adjacent matrix and adjacent lists use $\Theta(n^2)$ space.
  - If $m = O(n)$, adjacent list outperforms adjacent matrix

- However, one cannot tell in $O(1)$ time whether two vertices are connected

# Adjacency List vs. Matrix

- **Adjacency List**
  - More compact than adjacency matrices if graph has few edges
  - Requires more time to find if an edge exists

- **Adjacency Matrix**
  - Always require $n^2$ space
    - This can waste a lot of space if the number of edges are sparse
  - Can quickly find if an edge exists
  - It’s a matrix, some algorithms can be solved by matrix computation!

# Path between Vertices

- A path is a sequence of vertices ($v_0$, $v_1$, $v_2$,… $v_k$) such that:
  - For $0 \leq i < k$, $\{v_i, v_{i+1}\}$ is an edge
  - For $0 \leq i < k-1$, $v_i \neq v_{i+2}$

    *That is, the edge* $\{v_i, v_{i+1}\} \neq \{v_{i+1}, v_{i+2}\}$

  *Note: a path is allowed to go through the same vertex or the same edge any number of times!*

- The length of a path is the number of edges on the path

# Types of paths

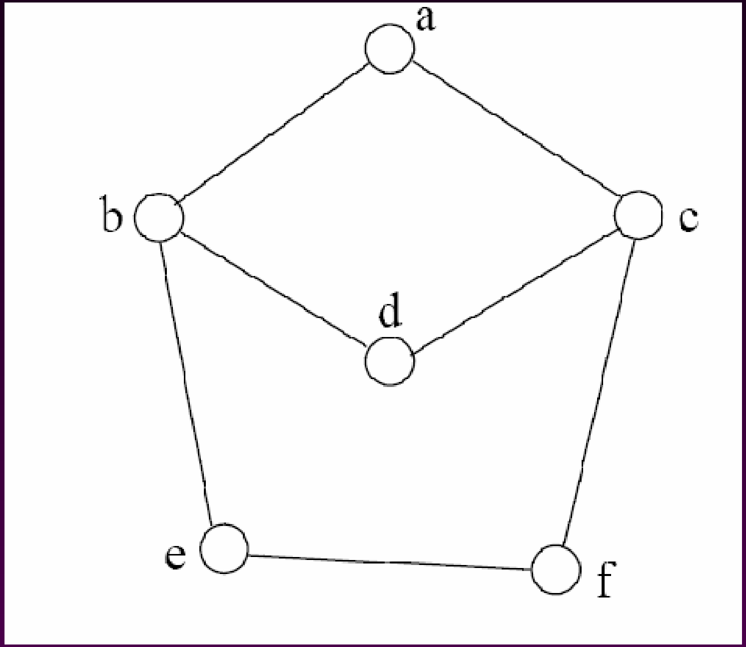
- A path is **simple** if and only if it does not contain a vertex more than once.
- A path is a **cycle** if and only if $v_0 = v_k$
  - The beginning and end are the same vertex!
- A path contains a cycle as its sub-path if some vertex appears twice or more

# Path Examples

Are these paths?

Any cycles?

What is the path’s length?



1. {a,c,f,e}
2. {a,b,d,c,f,e}
3. {a, c, d, b, d, c, f, e}
4. {a,c,d,b,a}
5. {a,c,f,e,b,d,c,a}

# Summary

- A graph G=(V, E) consists a set of vertices, V, and a set of edges, E. Each edge is a pair of *(v, w)*, where v, w belongs to V

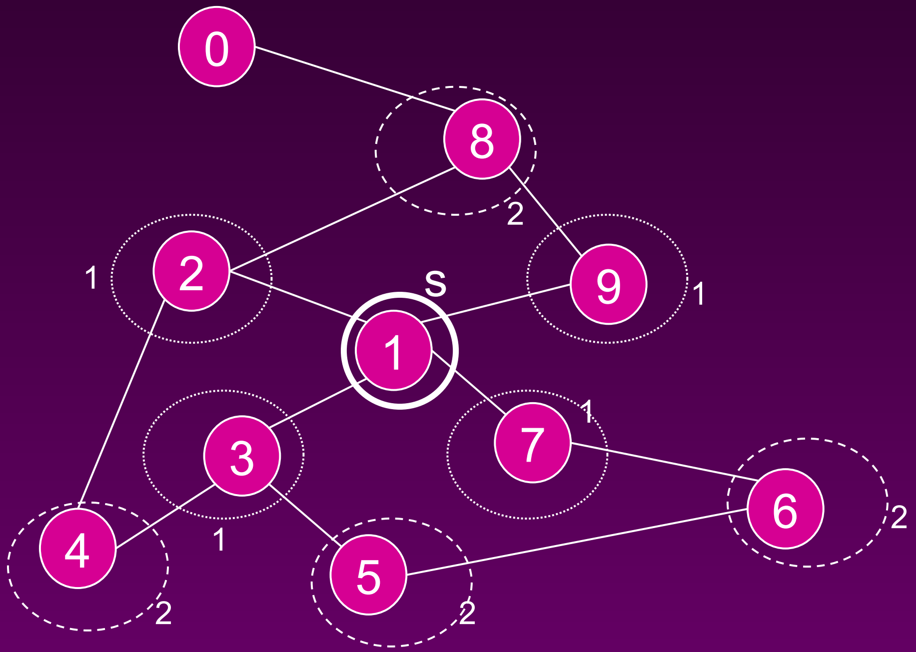
- graph, directed and undirected graph
- vertex, node, edge, arc
- incident, adjacent
- degree, in-degree, out-degree, isolated
- path, simple path,
- path of length k, subpath
- cycle, simple cycle, acyclic
- connected, connected component
- neighbor, complete graph, planar graph

# Graph Traversal

- Application example
  - Given a graph representation and a vertex **s** in the graph
  - Find all paths from **s** to other vertices
- Two common graph traversal algorithms
  - Breadth-First Search (BFS)
    - Find the shortest paths in an unweighted graph
  - Depth-First Search (DFS)
    - Topological sort
    - Find strongly connected components

# BFS and Shortest Path Problem

- Given any source vertex ***s***, BFS visits the other vertices at increasing distances away from s. In doing so, BFS discovers paths from s to other vertices
- What do we mean by "distance"? The number of edges on a path from s
- From 'local' to 'global', step by step.



Example

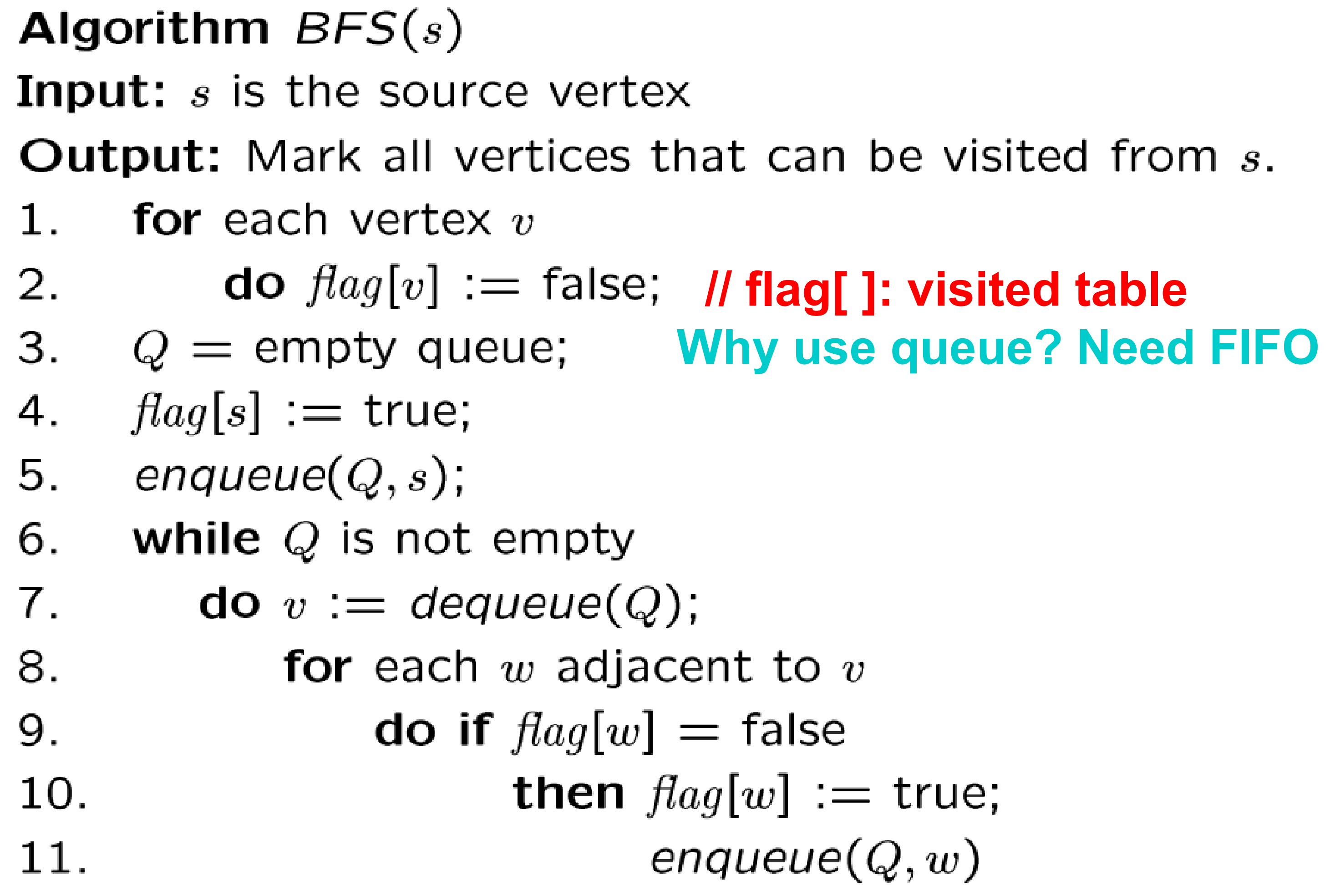
Consider s=vertex 1

Nodes at distance 1?
2, 3, 7, 9

Nodes at distance 2?
8, 6, 5, 4

Nodes at distance 3?
0

# BFS Algorithm

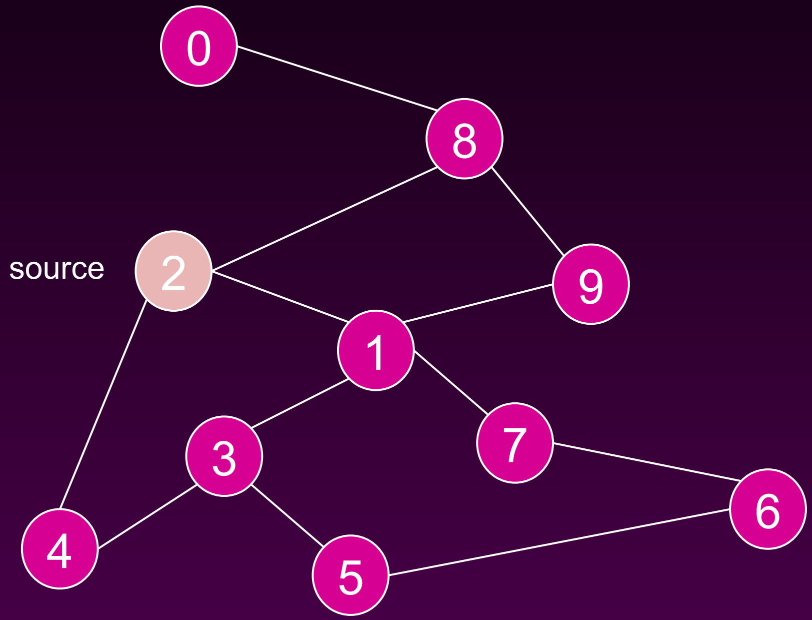


```
Algorithm BFS($s$)
Input: $s$ is the source vertex
Output: Mark all vertices that can be visited from $s$.
for each vertex $v$
    do $flag[v]$ := false;
$Q$ = empty queue;
$flag[s]$ := true;
enqueue($Q, s$);
while $Q$ is not empty
    do $v$ := dequeue($Q$);
        for each $w$ adjacent to $v$
            do if $flag[w]$ = false
                then $flag[w]$ := true;
                    enqueue($Q, w$)
```

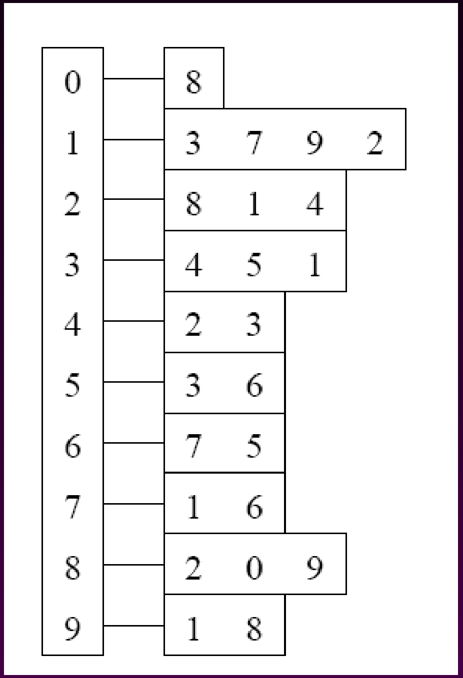
**// flag[ ]: visited table**

**Why use queue? Need FIFO**

# BFS Example



Adjacency List



0 → 8

1 → 3 7 9 2

2 → 8 1 4

3 → 4 5 1

4 → 2 3

5 → 3 6

6 → 7 5

7 → 1 6

8 → 2 0 9

9 → 1 8

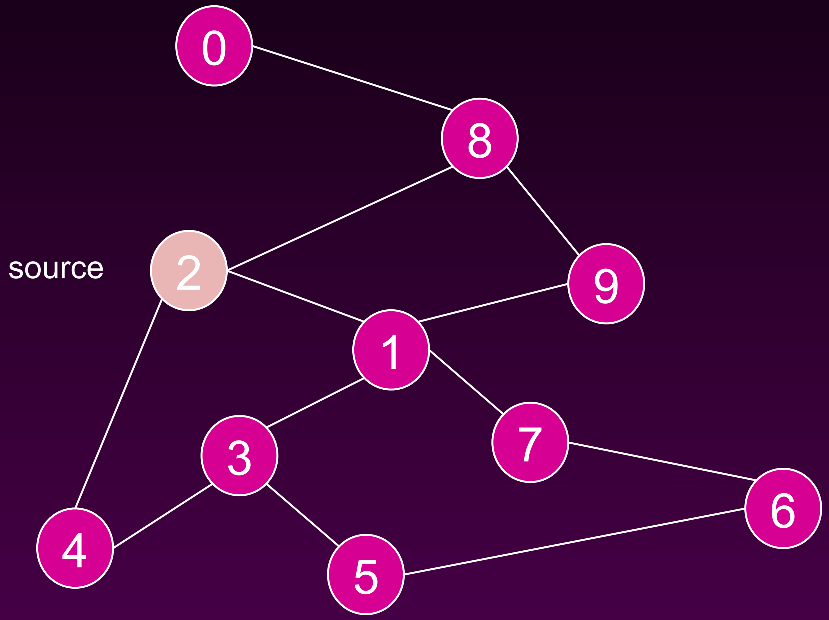
Visited Table (T/F)

| | |
|---|---|
| 0 | F |
| 1 | F |
| 2 | F |
| 3 | F |
| 4 | F |
| 5 | F |
| 6 | F |
| 7 | F |
| 8 | F |
| 9 | F |

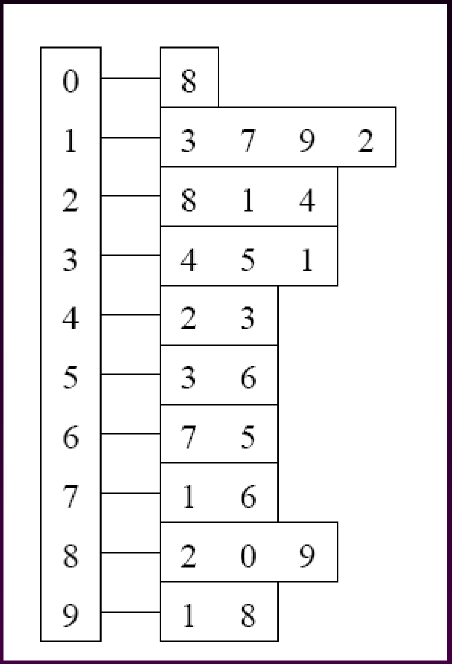
Initialize visited table (all False)

**Q =** { }

Initialize **Q** to be empty



## Adjacency List



| | | | | |
|---|---|---|---|---|
| 0 | 8 | | | |
| 1 | 3 | 7 | 9 | 2 |
| 2 | 8 | 1 | 4 | |
| 3 | 4 | 5 | 1 | |
| 4 | 2 | 3 | | |
| 5 | 3 | 6 | | |
| 6 | 7 | 5 | | |
| 7 | 1 | 6 | | |
| 8 | 2 | 0 | 9 | |
| 9 | 1 | 8 | | |

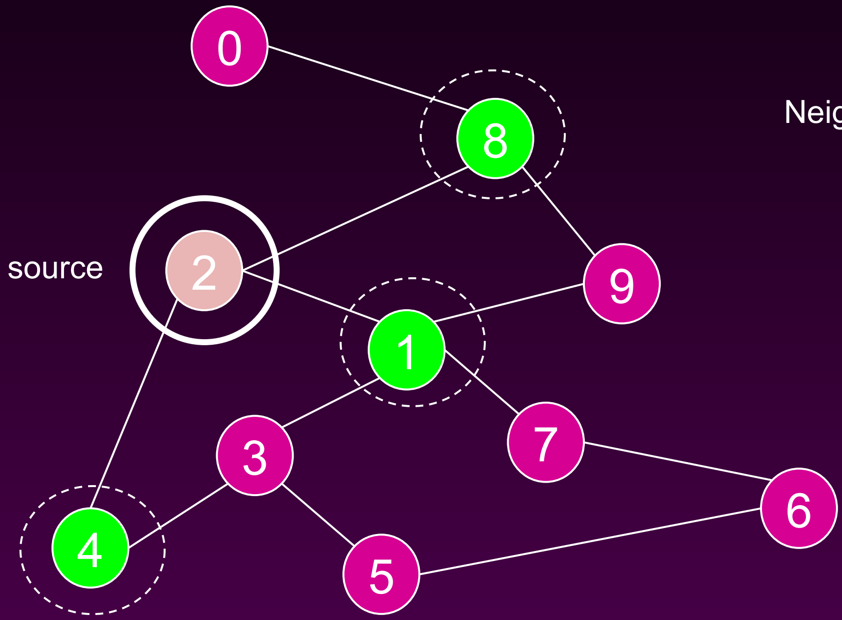
## Visited Table (T/F)

| | |
|---|---|
| 0 | F |
| 1 | F |
| 2 | T |
| 3 | F |
| 4 | F |
| 5 | F |
| 6 | F |
| 7 | F |
| 8 | F |
| 9 | F |

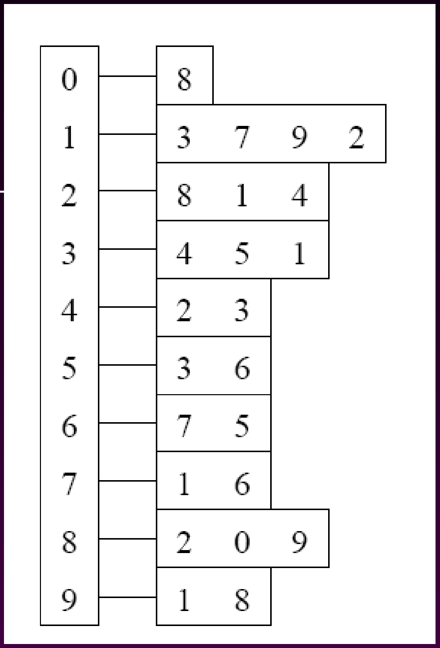
Flag that 2 has been visited

Q = { 2 }

Place source 2 on the queue



Adjacency List



0 — 8
1 — 3 7 9 2
2 — 8 1 4
3 — 4 5 1
4 — 2 3
5 — 3 6
6 — 7 5
7 — 1 6
8 — 2 0 9
9 — 1 8

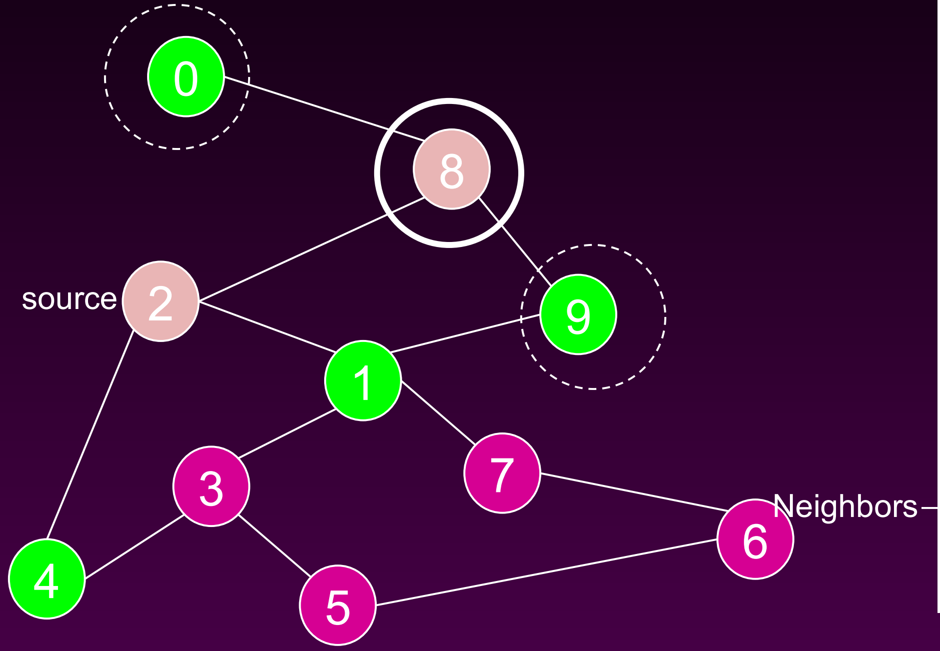
Visited Table (T/F)

| | |
|---|---|
| 0 | F |
| 1 | T |
| 2 | T |
| 3 | F |
| 4 | T |
| 5 | F |
| 6 | F |
| 7 | F |
| 8 | T |
| 9 | F |

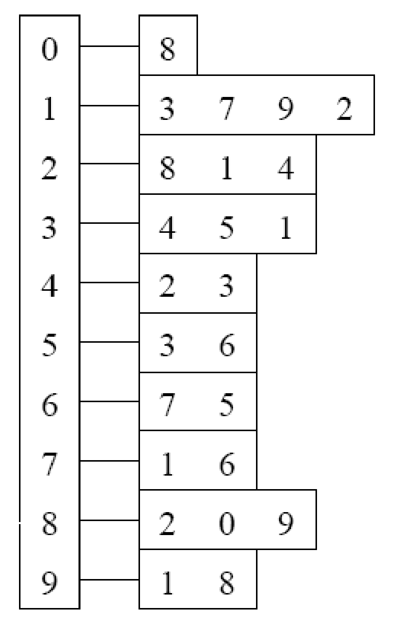
Mark neighbors as visited 1, 4, 8

**Q =** {2} → { 8, 1, 4 }

Dequeue 2.
Place all unvisited neighbors of 2 on the queue



## Adjacency List



| | |
|---|---|
| 0 | 8 |
| 1 | 3 7 9 2 |
| 2 | 8 1 4 |
| 3 | 4 5 1 |
| 4 | 2 3 |
| 5 | 3 6 |
| 6 | 7 5 |
| 7 | 1 6 |
| 8 | 2 0 9 |
| 9 | 1 8 |

## Visited Table (T/F)

| | |
|---|---|
| 0 | T |
| 1 | T |
| 2 | T |
| 3 | F |
| 4 | T |
| 5 | F |
| 6 | F |
| 7 | F |
| 8 | T |
| 9 | T |

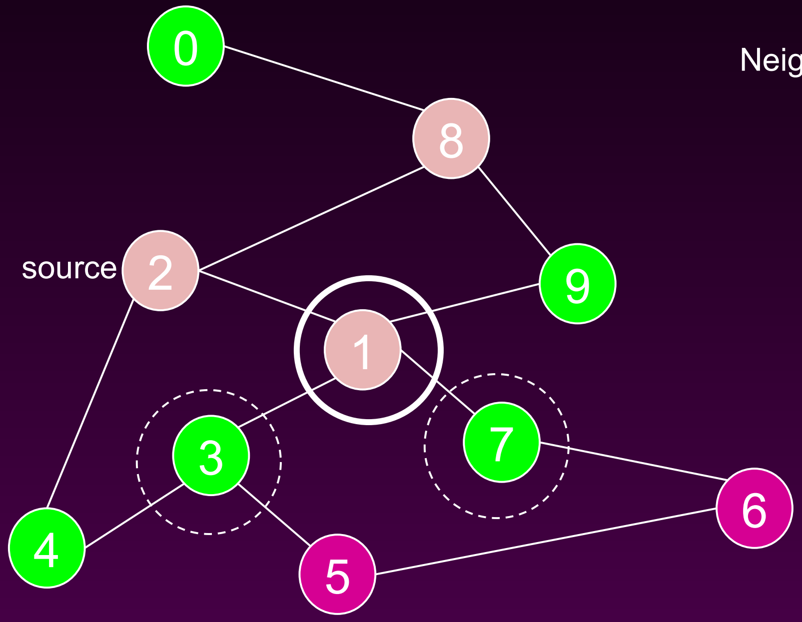
Mark new visited
Neighbors 0, 9

**Q =** { 8, 1, 4 } → { 1, 4, 0, 9 }

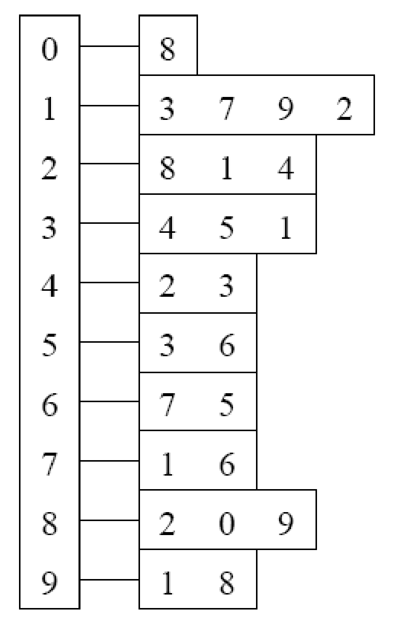
Dequeue 8.

-- Place all unvisited neighbors of 8 on the queue.

-- Notice that 2 is not placed on the queue again, it has been visited!



## Adjacency List



| | |
|---|---|
| 0 | 8 |
| 1 | 3 7 9 2 |
| 2 | 8 1 4 |
| 3 | 4 5 1 |
| 4 | 2 3 |
| 5 | 3 6 |
| 6 | 7 5 |
| 7 | 1 6 |
| 8 | 2 0 9 |
| 9 | 1 8 |

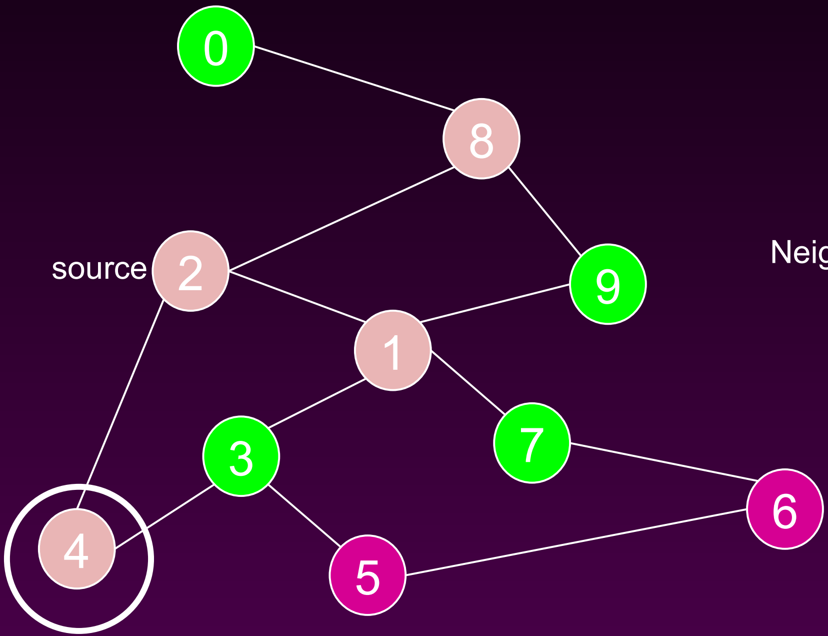
## Visited Table (T/F)

| | |
|---|---|
| 0 | T |
| 1 | T |
| 2 | T |
| 3 | T |
| 4 | T |
| 5 | F |
| 6 | F |
| 7 | T |
| 8 | T |
| 9 | T |

Mark new visited
Neighbors 3, 7

**Q =** { 1, 4, 0, 9 } → { 4, 0, 9, 3, 7 }

Dequeue 1.
-- Place all unvisited neighbors of 1 on the queue.
-- Only nodes 3 and 7 haven’t been visited yet.



## Adjacency List

Neighbors

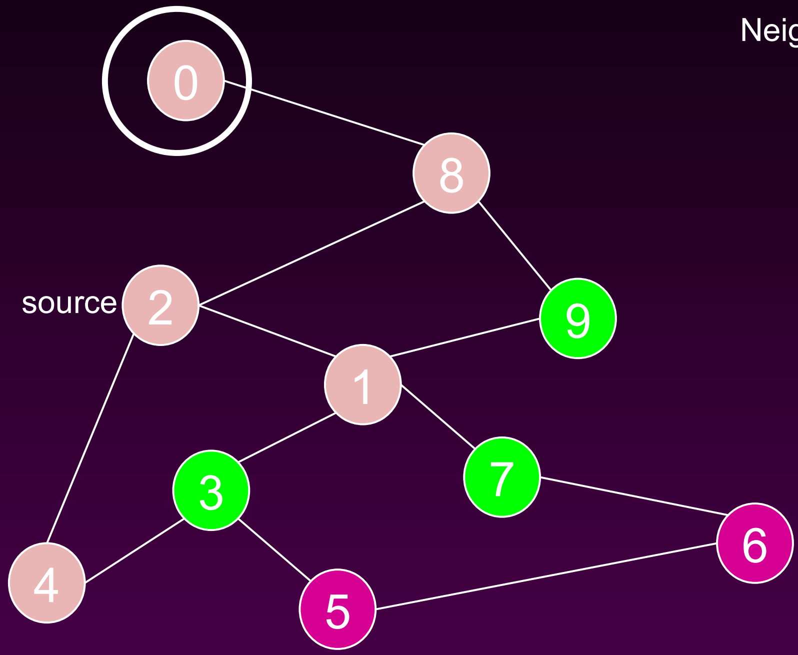
| | |
|---|---|
| 0 | 8 |
| 1 | 3 7 9 2 |
| 2 | 8 1 4 |
| 3 | 4 5 1 |
| 4 | 2 3 |
| 5 | 3 6 |
| 6 | 7 5 |
| 7 | 1 6 |
| 8 | 2 0 9 |
| 9 | 1 8 |

## Visited Table (T/F)

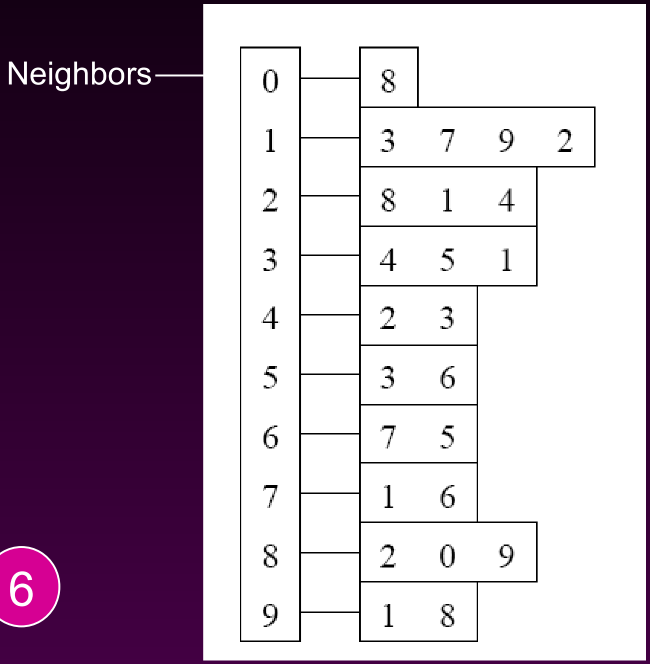
| | |
|---|---|
| 0 | T |
| 1 | T |
| 2 | T |
| 3 | T |
| 4 | T |
| 5 | F |
| 6 | F |
| 7 | T |
| 8 | T |
| 9 | T |

**Q =** { 4, 0, 9, 3, 7 } → { 0, 9, 3, 7 }

Dequeue 4.
-- 4 has no unvisited neighbors!



## Adjacency List



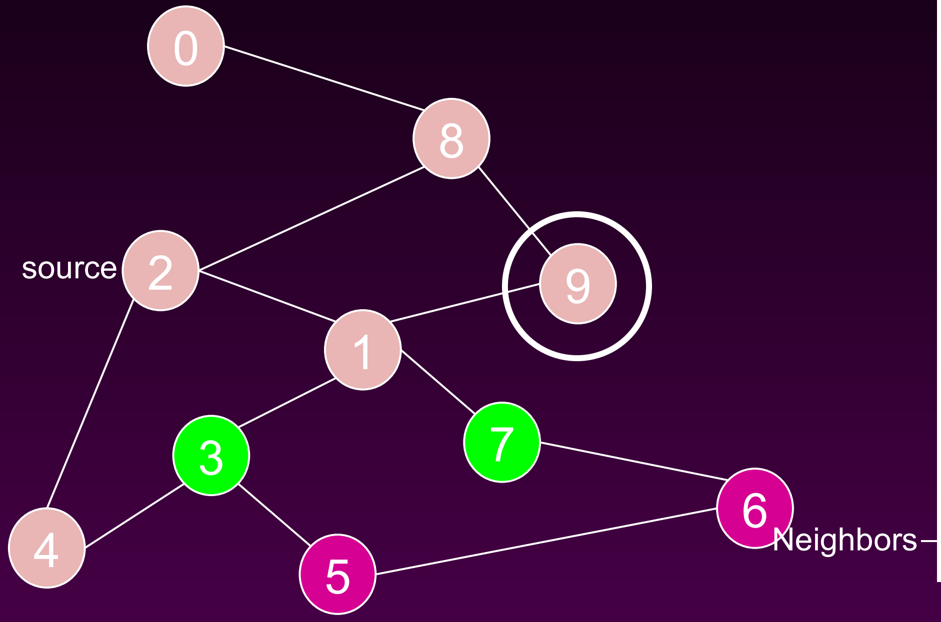
| | |
|---|---|
| 0 | 8 |
| 1 | 3 7 9 2 |
| 2 | 8 1 4 |
| 3 | 4 5 1 |
| 4 | 2 3 |
| 5 | 3 6 |
| 6 | 7 5 |
| 7 | 1 6 |
| 8 | 2 0 9 |
| 9 | 1 8 |

## Visited Table (T/F)

| | |
|---|---|
| 0 | T |
| 1 | T |
| 2 | T |
| 3 | T |
| 4 | T |
| 5 | F |
| 6 | F |
| 7 | T |
| 8 | T |
| 9 | T |

**Q =** { 0, 9, 3, 7 } → { 9, 3, 7 }

Dequeue 0.
-- 0 has no unvisited neighbors!



## Adjacency List

| | | | | |
|---|---|---|---|---|
| 0 | 8 | | | |
| 1 | 3 | 7 | 9 | 2 |
| 2 | 8 | 1 | 4 | |
| 3 | 4 | 5 | 1 | |
| 4 | 2 | 3 | | |
| 5 | 3 | 6 | | |
| 6 | 7 | 5 | | |
| 7 | 1 | 6 | | |
| 8 | 2 | 0 | 9 | |
| 9 | 1 | 8 | | |

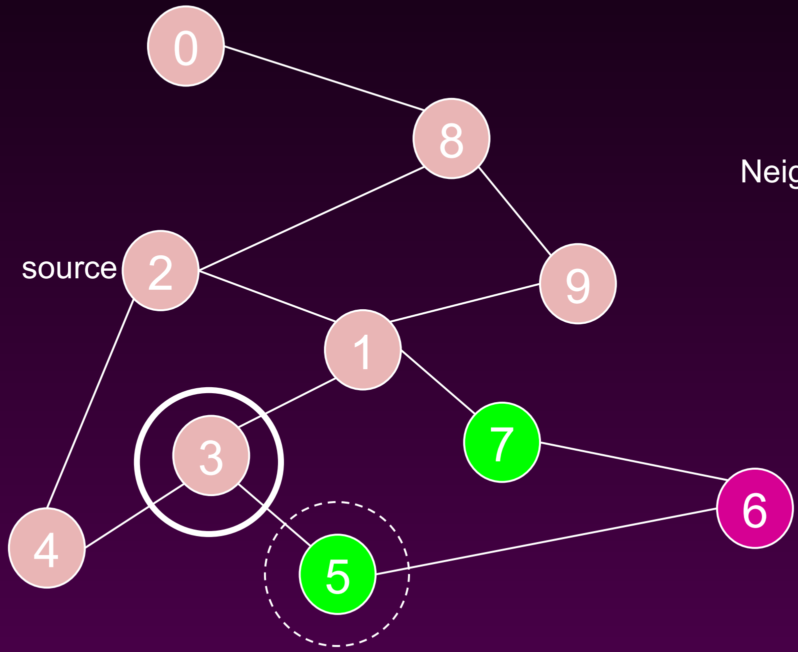
## Visited Table (T/F)

| | |
|---|---|
| 0 | T |
| 1 | T |
| 2 | T |
| 3 | T |
| 4 | T |
| 5 | F |
| 6 | F |
| 7 | T |
| 8 | T |
| 9 | T |

**Q =** { 9, 3, 7 } → { 3, 7 }

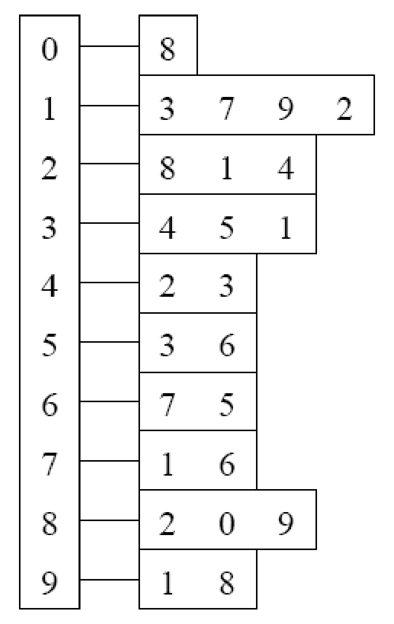
Dequeue 9.

-- 9 has no unvisited neighbors!



## Adjacency List

Neighbors



| | |
|---|---|
| 0 | 8 |
| 1 | 3 7 9 2 |
| 2 | 8 1 4 |
| 3 | 4 5 1 |
| 4 | 2 3 |
| 5 | 3 6 |
| 6 | 7 5 |
| 7 | 1 6 |
| 8 | 2 0 9 |
| 9 | 1 8 |

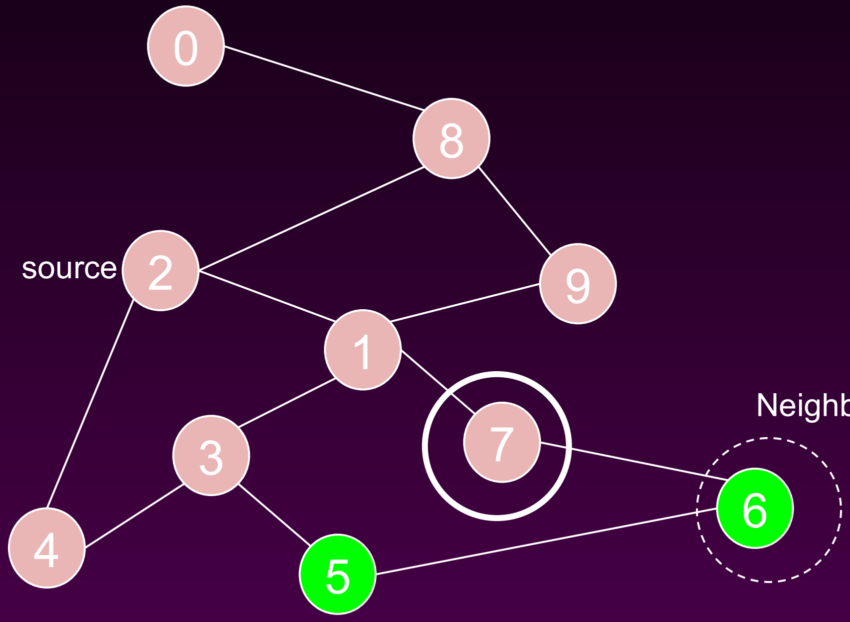
## Visited Table (T/F)

| | |
|---|---|
| 0 | T |
| 1 | T |
| 2 | T |
| 3 | T |
| 4 | T |
| 5 | T |
| 6 | F |
| 7 | T |
| 8 | T |
| 9 | T |

Mark new visited
Vertex 5

**Q =** { 3, 7 } → { 7, 5 }

Dequeue 3.
-- place neighbor 5 on the queue.



## Adjacency List

Neighbors

| | |
|---|---|
| 0 | 8 |
| 1 | 3 7 9 2 |
| 2 | 8 1 4 |
| 3 | 4 5 1 |
| 4 | 2 3 |
| 5 | 3 6 |
| 6 | 7 5 |
| 7 | 1 6 |
| 8 | 2 0 9 |
| 9 | 1 8 |

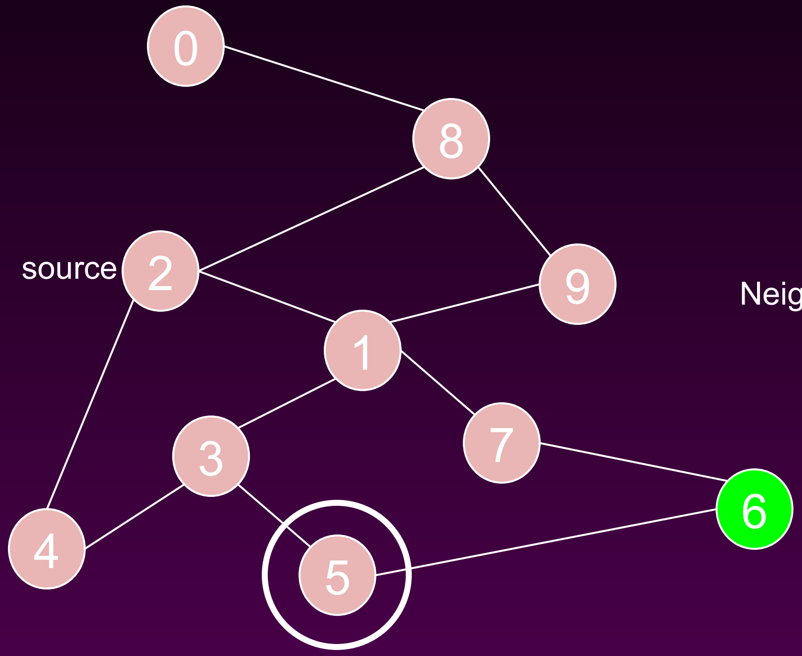
## Visited Table (T/F)

| | |
|---|---|
| **0** | T |
| **1** | T |
| **2** | T |
| **3** | T |
| **4** | T |
| **5** | T |
| **6** | T |
| **7** | T |
| **8** | T |
| **9** | T |

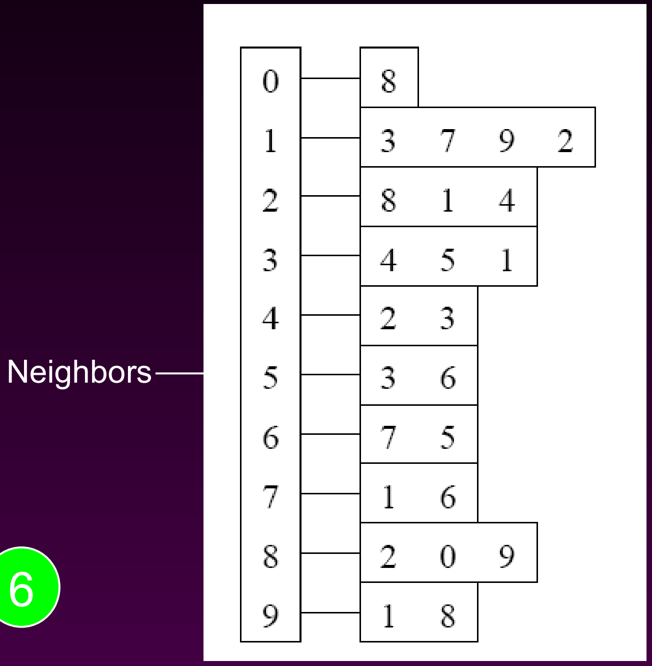
Mark new visited
Vertex 6

**Q =** { 7, 5 } → { 5, 6 }

Dequeue 7.
-- place neighbor 6 on the queue



## Adjacency List



| | | | | |
|---|---|---|---|---|
| 0 | 8 | | | |
| 1 | 3 | 7 | 9 | 2 |
| 2 | 8 | 1 | 4 | |
| 3 | 4 | 5 | 1 | |
| 4 | 2 | 3 | | |
| 5 | 3 | 6 | | |
| 6 | 7 | 5 | | |
| 7 | 1 | 6 | | |
| 8 | 2 | 0 | 9 | |
| 9 | 1 | 8 | | |

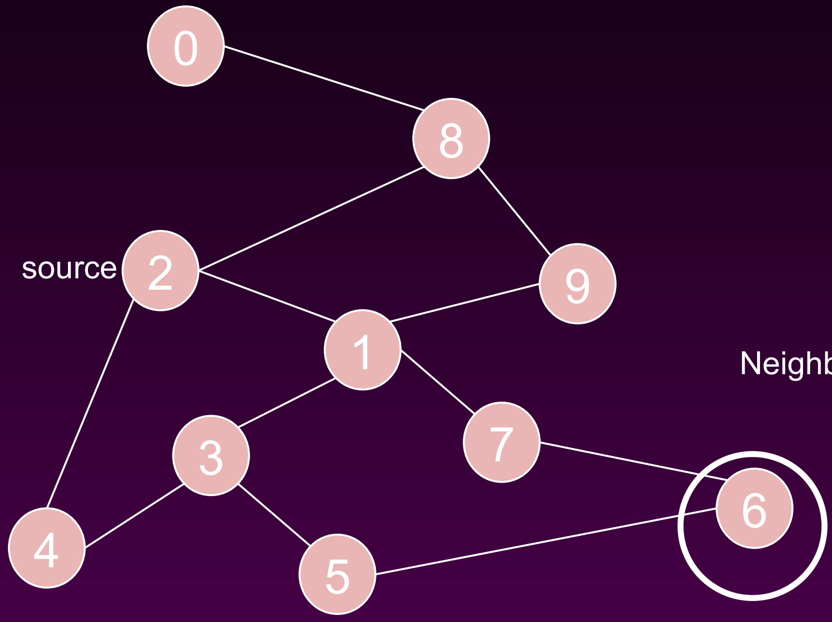
## Visited Table (T/F)

| | |
|---|---|
| 0 | T |
| 1 | T |
| 2 | T |
| 3 | T |
| 4 | T |
| 5 | T |
| 6 | T |
| 7 | T |
| 8 | T |
| 9 | T |

**Q =** { 5, 6} → { 6 }

Dequeue 5.

-- no unvisited neighbors of 5



## Adjacency List

Neighbors

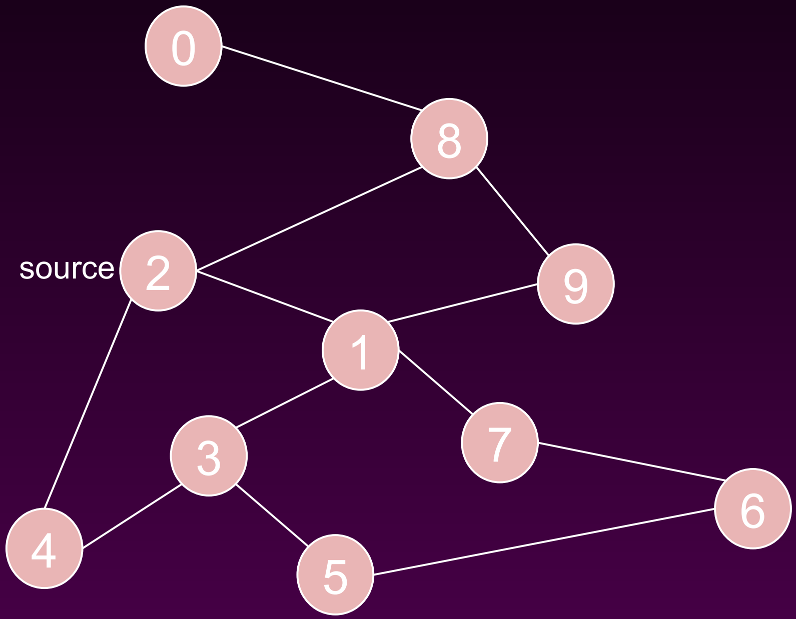
| | |
|---|---|
| 0 | 8 |
| 1 | 3 7 9 2 |
| 2 | 8 1 4 |
| 3 | 4 5 1 |
| 4 | 2 3 |
| 5 | 3 6 |
| 6 | 7 5 |
| 7 | 1 6 |
| 8 | 2 0 9 |
| 9 | 1 8 |

## Visited Table (T/F)

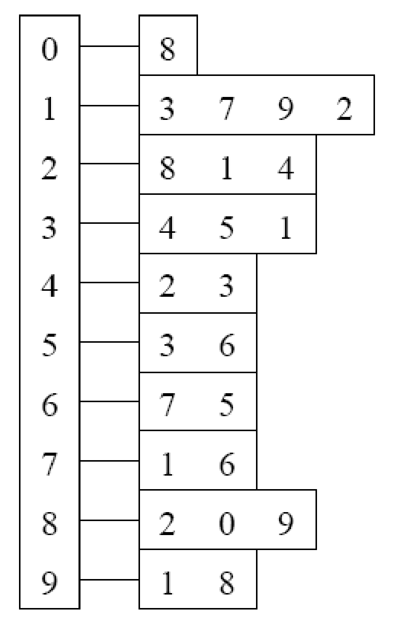
| | |
|---|---|
| 0 | T |
| 1 | T |
| 2 | T |
| 3 | T |
| 4 | T |
| 5 | T |
| 6 | T |
| 7 | T |
| 8 | T |
| 9 | T |

**Q =** { 6 } → { }

Dequeue 6.
-- no unvisited neighbors of 6



Adjacency List



| | | | | |
|---|---|---|---|---|
| 0 | 8 | | | |
| 1 | 3 | 7 | 9 | 2 |
| 2 | 8 | 1 | 4 | |
| 3 | 4 | 5 | 1 | |
| 4 | 2 | 3 | | |
| 5 | 3 | 6 | | |
| 6 | 7 | 5 | | |
| 7 | 1 | 6 | | |
| 8 | 2 | 0 | 9 | |
| 9 | 1 | 8 | | |

Visited Table (T/F)

| | |
|---|---|
| 0 | T |
| 1 | T |
| 2 | T |
| 3 | T |
| 4 | T |
| 5 | T |
| 6 | T |
| 7 | T |
| 8 | T |
| 9 | T |

**Q = { } STOP!!! Q is empty!!!**

What did we discover?

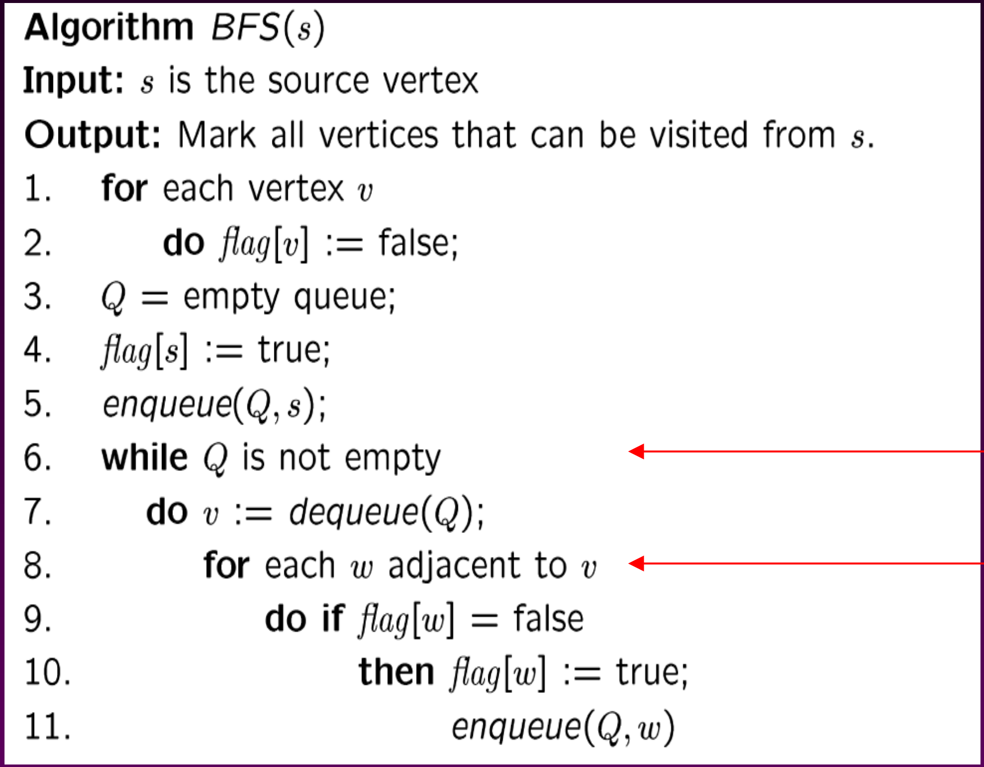
Look at "visited" tables.

There exists a path from source vertex 2 to all vertices in the graph

# Time Complexity of BFS

## (Using Adjacency List)

- Assume adjacency list
  - n = number of vertices   m = number of edges



```
Algorithm BFS($s$)
Input: $s$ is the source vertex
Output: Mark all vertices that can be visited from $s$.
for each vertex $v$
    do $flag[v]$ := false;
$Q$ = empty queue;
$flag[s]$ := true;
$enqueue(Q, s)$;
while $Q$ is not empty
    do $v := dequeue(Q)$;
        for each $w$ adjacent to $v$
            do if $flag[w]$ = false
                then $flag[w]$ := true;
                    $enqueue(Q, w)$
```

**O(n + m)**

Each vertex will enter Q at most once.

Each iteration takes time proportional to deg(v) + 1  (the number 1 is to account for the case where deg(v) = 0 --- the work required is 1, not 0).

# Running Time

- Recall: Given a graph with m edges, what is the total degree?

  $$\Sigma_{\text{vertex } v} \ \deg(v) = 2m$$

- The total running time of the while loop is:

  $$O(\ \Sigma_{\text{vertex } v} \ (\deg(v) + 1)\ ) = O(n+m)$$

  this is summing over all the iterations in the while loop!

# Time Complexity of BFS

## (Using Adjacency Matrix)

- Assume adjacency list
  - n = number of vertices   m = number of edges

```
Algorithm BFS($s$)
Input: $s$ is the source vertex
Output: Mark all vertices that can be visited from $s$.
for each vertex $v$
    do $flag[v]$ := false;
$Q$ = empty queue;
$flag[s]$ := true;
enqueue($Q, s$);
while $Q$ is not empty
    do $v$ := dequeue($Q$);
        for each $w$ adjacent to $v$
            do if $flag[w]$ = false
                then $flag[w]$ := true;
                    enqueue($Q, w$)
```

**$O(n^2)$**

Finding the adjacent vertices of v requires checking all elements in the row. This takes linear time O(n).

Summing over all the n iterations, the total running time is $O(n^2)$.

So, with adjacency matrix, BFS is $O(n^2)$ independent of the number of edges m. With adjacent lists, BFS is O(n+m); if $m=O(n^2)$ like in a dense graph, $O(n+m)=O(n^2)$.