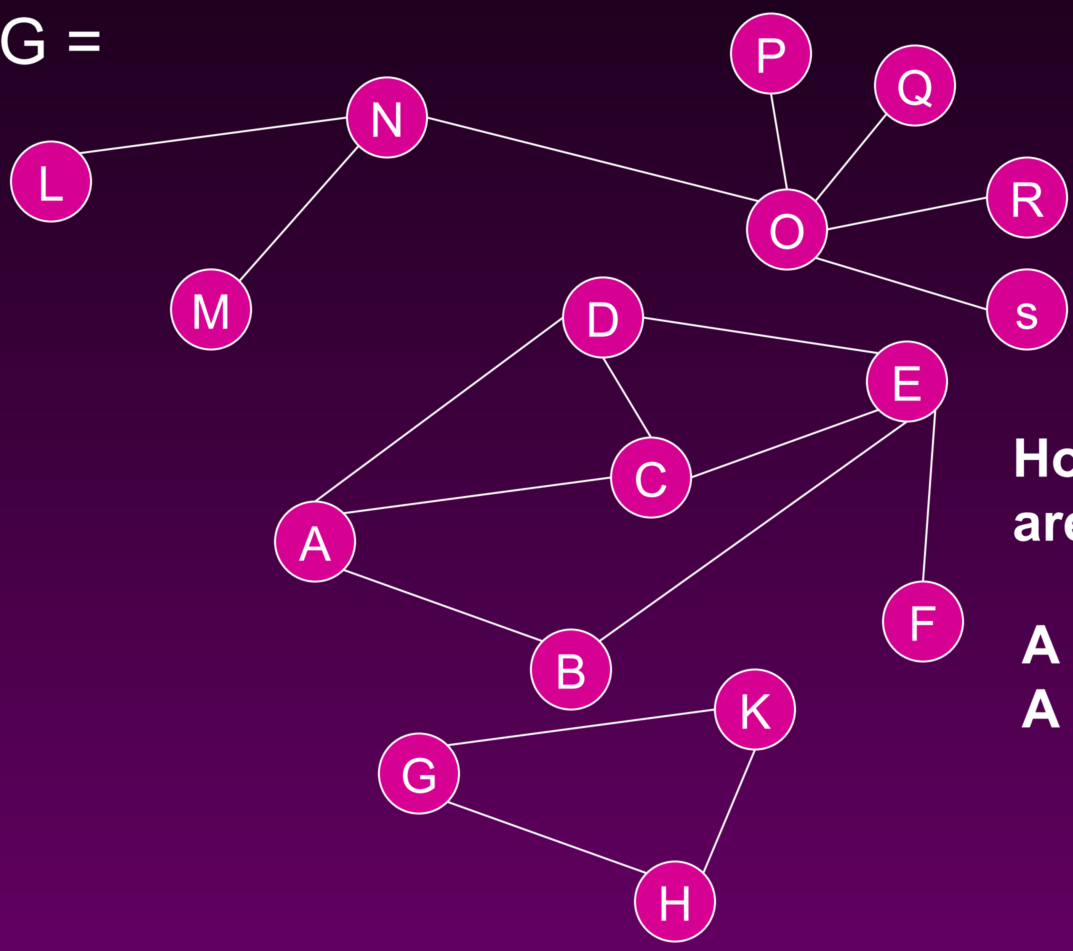
Connected Components, Directed Graphs, Topological Sort

Graph Application: Connectivity

G =

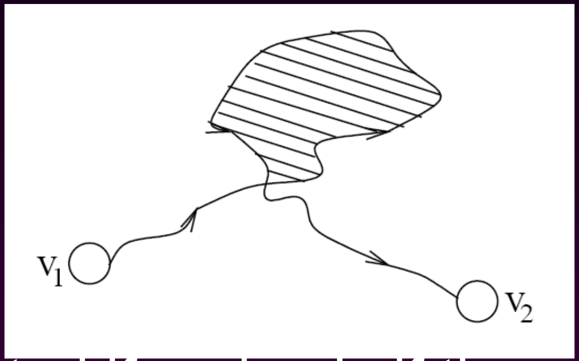


How do we tell if two vertices are connected?

A connected to F?
A connected to L?

Connectivity

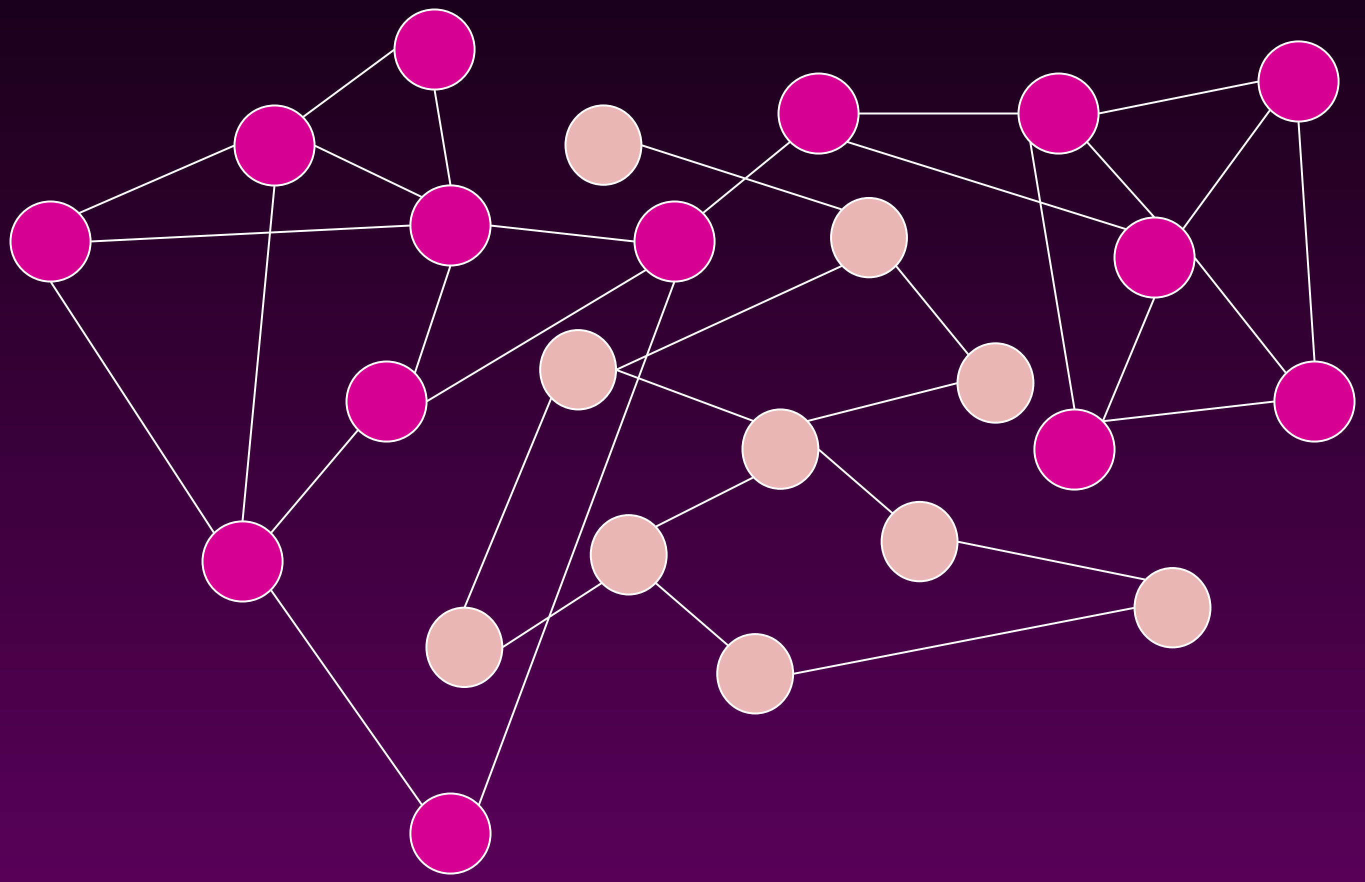
- A graph is *connected* if and only if there exists a path between every pair of distinct vertices.



- A graph is connected if and only if there exists a simple path between every pair of distinct vertices
  - since every non-simple path contains a cycle, which can be bypassed

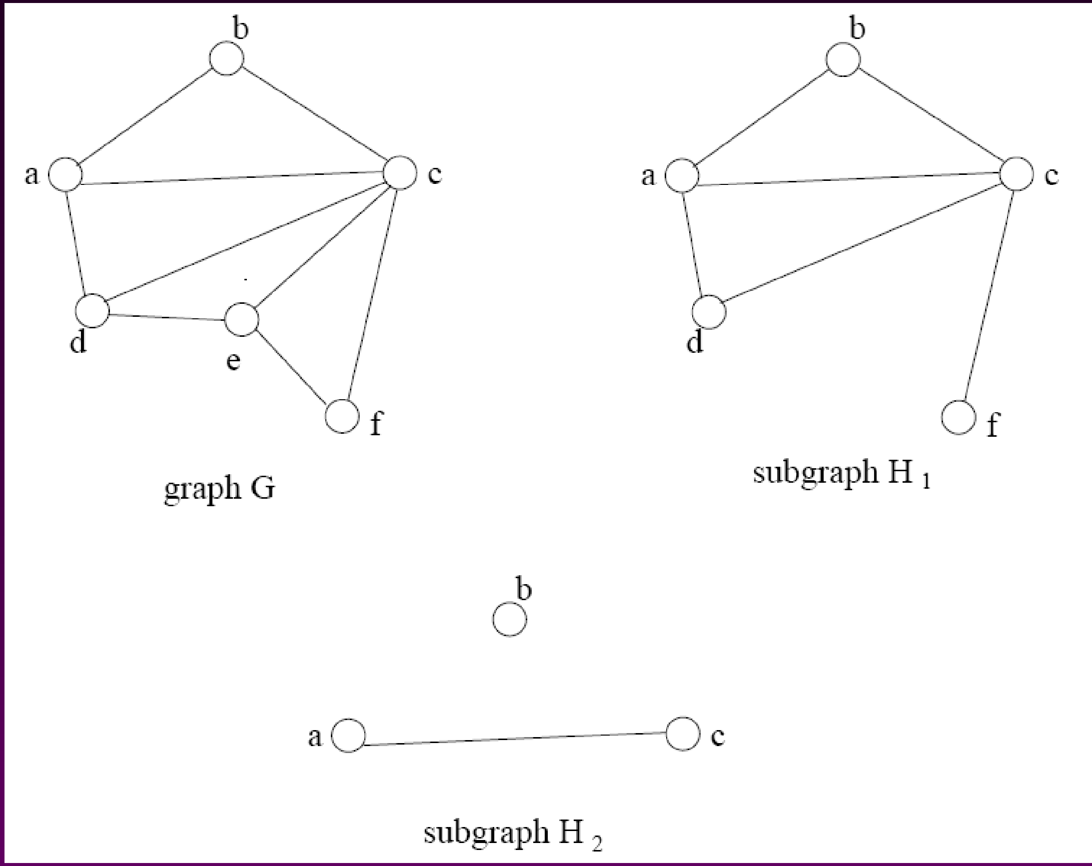
- How to check for connectivity?
  - Run BFS or DFS (using an arbitrary vertex as the source)
  - If all vertices have been visited, the graph is connected.
  - Running time? O(n + m)

Connected Components



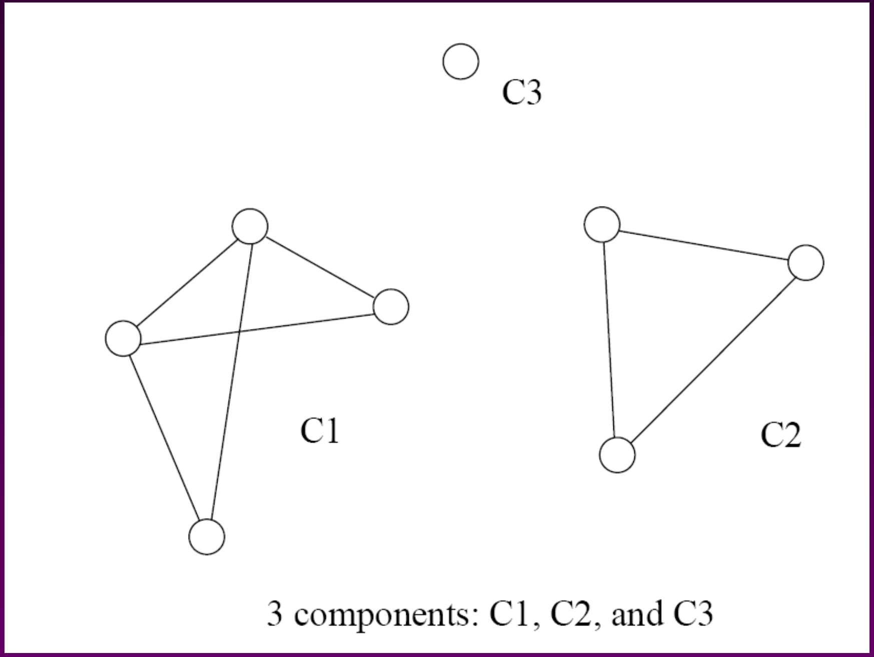
Subgraphs

A graph $H(V_H, E_H)$ is a *subgraph* of $G(V_G, E_G)$ if and only if $V_H \subset V_G$ and $E_H \subset E_G$.



Connected Components

- Formal definition
  - A connected component is a **maximal** connected subgraph of a graph
- The set of connected components is unique for a given graph



Finding Connected Components

Algorithm *DFSConn*(G)
Input: a graph G
Output: the connected components
1. **for** each vertex v
2. **do** $\mathit{flag}[v]$:= false;
3. $\boxed{\textbf{for} \text{ each vertex } v}$ For each vertex
4. $\boxed{\textbf{do if } \mathit{flag}[v] = \text{false}}$ If not visited
5. **then** output "A new connected component:";
6. $\boxed{\mathit{RDFS}(v);}$ Call DFS

This will find all vertices connected to “v” => one connected component

Algorithm *RDFS*(v)
1. $\mathit{flag}[v]$:= true;
2. output v;
3. **for** each neighbor w of v
4. **do if** $\mathit{flag}[w]$ = false
5. **then** *RDFS*(w);

Basic DFS algorithm

Time Complexity

- Running time for each *i* connected component

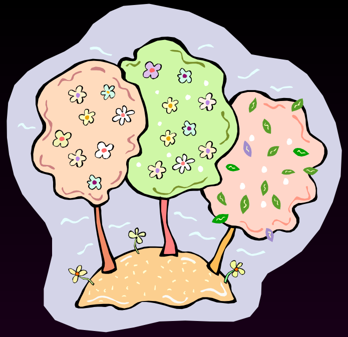
$$O(n_i + m_i)$$

- Running time for the graph G

$$\sum_i O(n_i + m_i) = O(\sum_i n_i + \sum_i m_i) = O(n + m)$$

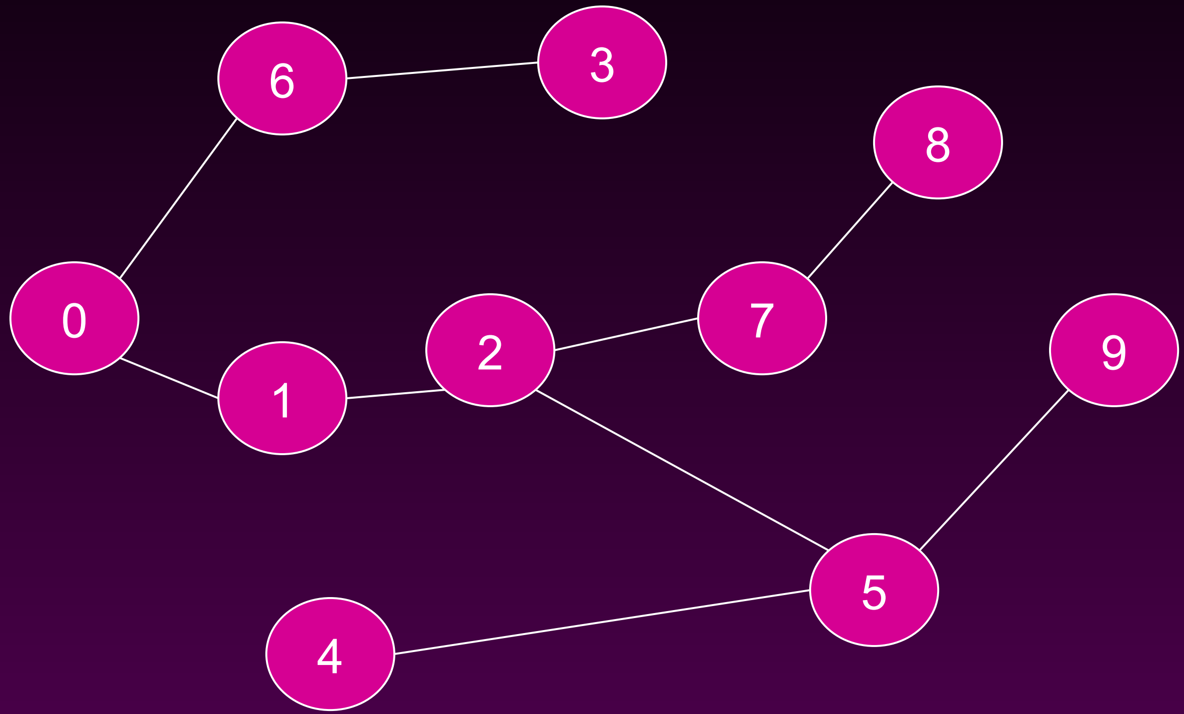
- Reason: Can two connected components share
  - the same edge?
  - the same vertex?

Trees



- Tree arises in many computer science applications
- A graph G is a tree if and only if it is connected and acyclic
  (Acyclic means it does not contain any simple cycles)
- The following statements are equivalent
  - G is a tree
  - G is acyclic and has exactly n-1 edges
  - G is connected and has exactly n-1 edges

Tree Example



- Is it a graph?
- Does it contain cycles? In other words, is it acyclic?
- How many vertices?
- How many edges?

Directed Graph

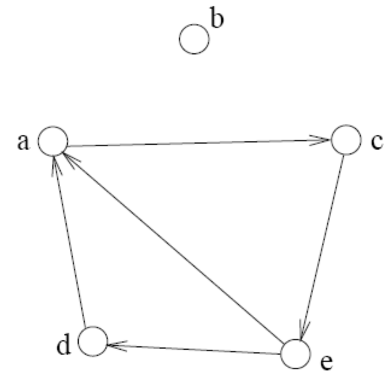
- A graph is directed if direction is assigned to each edge.
- Directed edges are denoted as *arcs*.
  - Arc is an ordered pair (u, v)

- Recall: for an undirected graph
  - An edge is denoted {u,v}, which actually corresponds to two arcs (u,v) and (v,u)

Representations

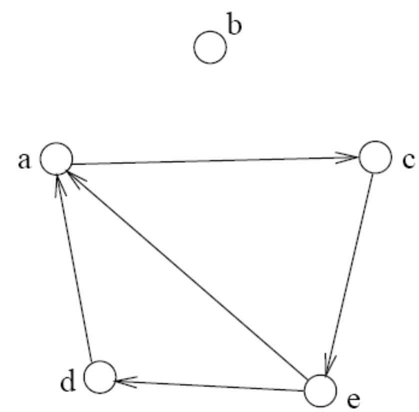
- The adjacency matrix and adjacency list can be used

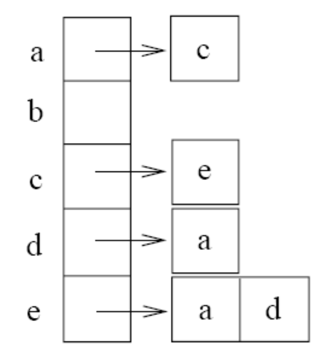
1. *Adjacency Matrix*



	a	b	c	d	e
a	0	0	1	0	0
b	0	0	0	0	0
c	0	0	0	0	1
d	1	0	0	0	0
e	1	0	0	1	0

2. *Adjacency List*





Directed Acyclic Graph

- A directed path is a sequence of vertices $(v_0, v_1, \ldots, v_k)$
  - Such that (v_i, v_{i+1}) is an *arc*

- *A* directed cycle is a directed path such that the first and last vertices are the same.

- A directed graph is acyclic if it does not contain any directed cycles

Indegree and Outdegree

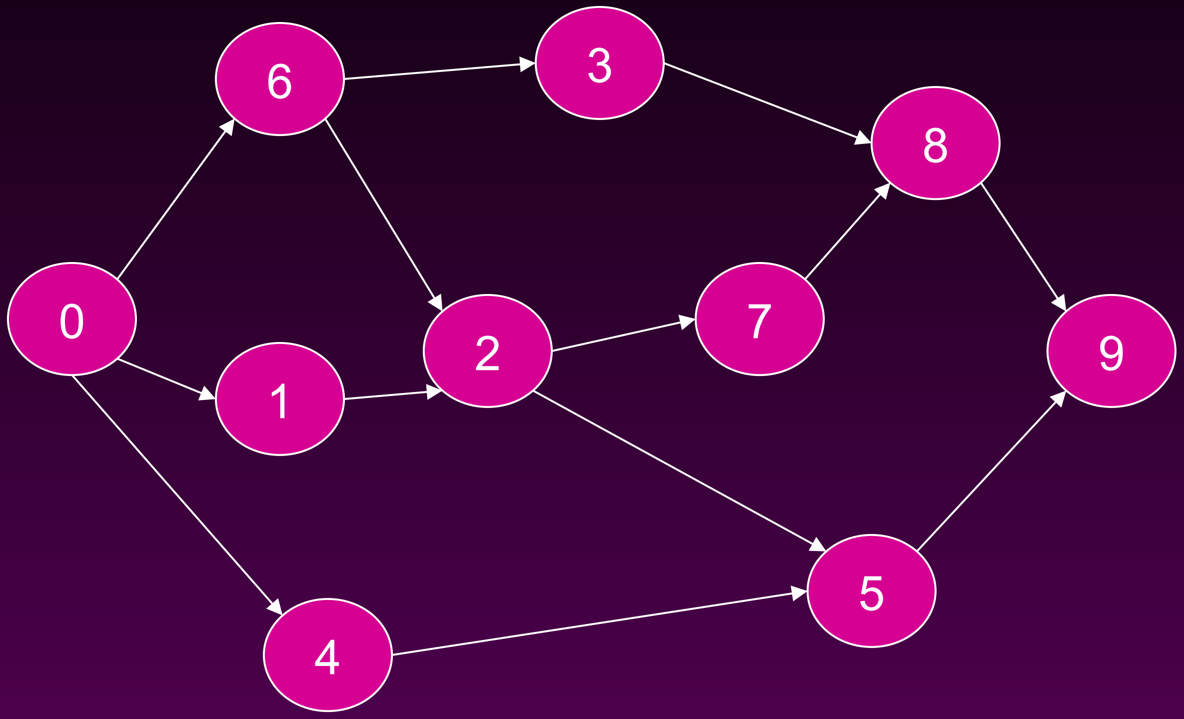
- Since the edges are directed
  - We can't simply talk about Deg(v)
- Instead, we need to consider the arcs coming "in" and going "out"
  - Thus, we define terms Indegree(v), and Outdegree(v)
- Each arc(u,v) contributes count 1 to the outdegree of u and the indegree of v

$$\sum_{\text{vertex } v} \text{indegree}(v) = \sum \text{outdegree}(v) = m$$

Calculate Indegree and Outdegree

- Outdegree is simple to compute
  - Scan through list Adj[v] and count the arcs
- Indegree calculation
  - First, initialize indegree[v]=0 for each vertex v
  - Scan through adj[v] list for each v
    - For each vertex w seen, indegree[w]++;
    - Running time: O(n+m)

Example



Indeg(2)?

Indeg(8)?

Outdeg(0)?

Num of Edges?

Total OutDeg?

Total Indeg?

Directed Graphs Usage

- Directed graphs are often used to represent order-dependent tasks
  - That is we cannot start a task before another task finishes

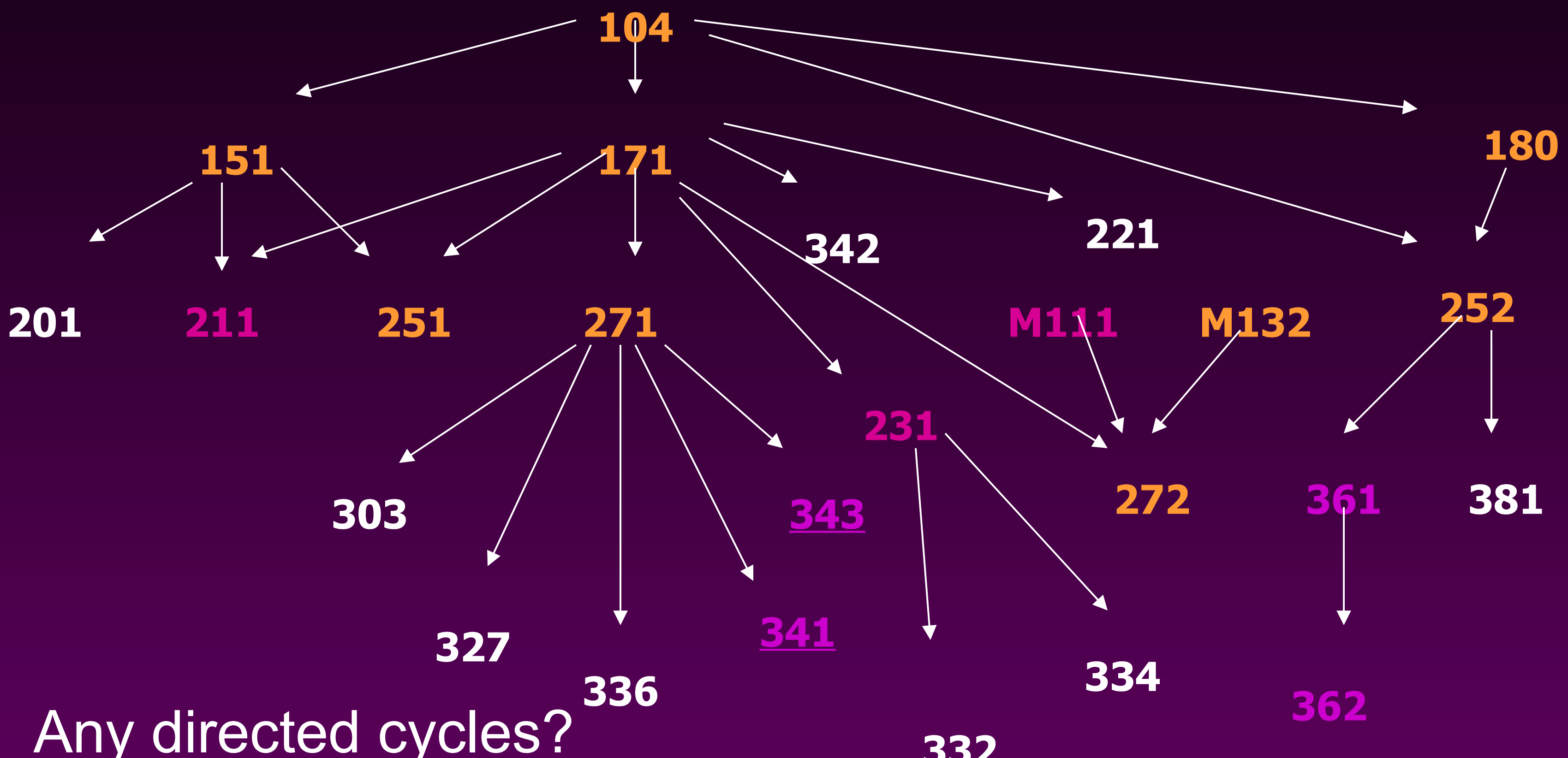
- We can model this task dependent constraint using *arcs*
- An *arc (i,j)* means *task j* cannot start until *task i* is finished

Task j cannot start
until task i is finished

- Clearly, for the system not to hang, the graph must be acyclic

University Example

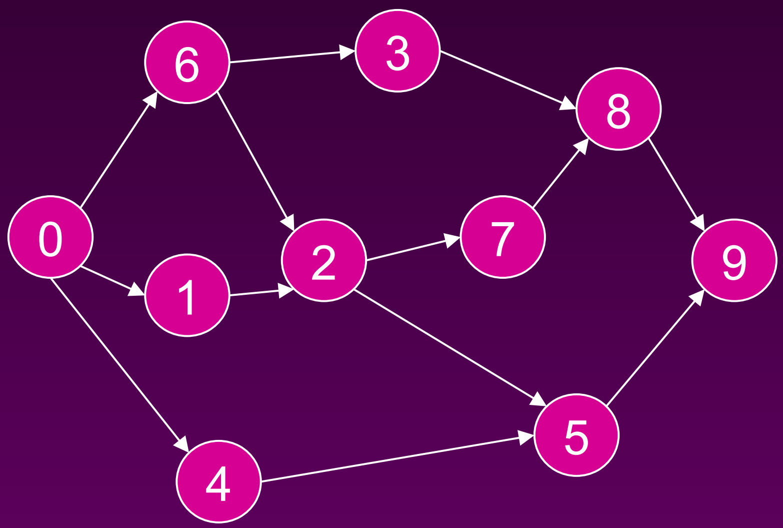
- CS departments course structure



Any directed cycles?
How many indeg(171)?
How many outdeg(171)?

Topological Sort

- Topological sort is an algorithm for a directed acyclic graph
- Linearly order the vertices so that the linear order respects the ordering relations implied by the arcs



For example:

0, 1, 2, 5, 9
0, 4, 5, 9
0, 6, 3, 7 ?

It may not be unique as they are many 'equal' elements!

Topological Sort Algorithm

- Observations
  - Starting point must have zero indegree
  - If it doesn't exist, the graph would not be acyclic
- Algorithm

1. A vertex with zero *indegree* is a task that can start right away. So we can output it first in the linear order
2. If a vertex *i* is output, then its outgoing arcs *(i, j)* are no longer useful, since tasks *j* does not need to wait for *i* anymore- so remove all *i*'s outgoing arcs
3. With vertex *i* removed, the new graph is still a directed acyclic graph. So, repeat step 1-2 until no vertex is left.

Topological Sort

Algorithm *TSort*(G)

Input: a directed acyclic graph G

Output: a topological ordering of vertices

```
initialize Q to be an empty queue;
for each vertex v
    do if indegree(v) = 0
        then enqueue(Q, v);
while Q is non-empty
    do v := dequeue(Q);
        output v;
        for each arc (v, w)
            do indegree(w) = indegree(w) − 1;
            if indegree(w) = 0
                then enqueue(w);
```

Find all starting points

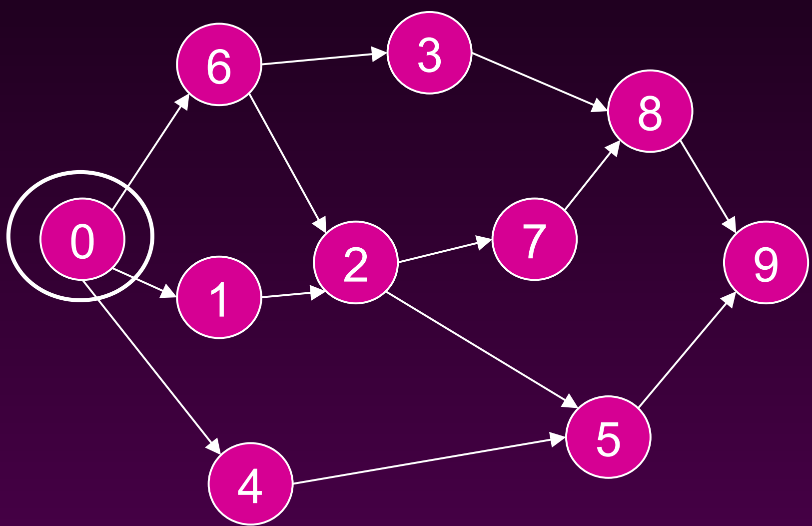
Take all outgoing arcs, all 'w's

Reduce indegree(w)

Place new start vertices on the Q

The running time is $O(n + m)$.

Example



0	6	1	4
1	2		
2	7	5	
3	8		
4	5		
5	9		
6	3	2	
7	8		
8	9		
9			

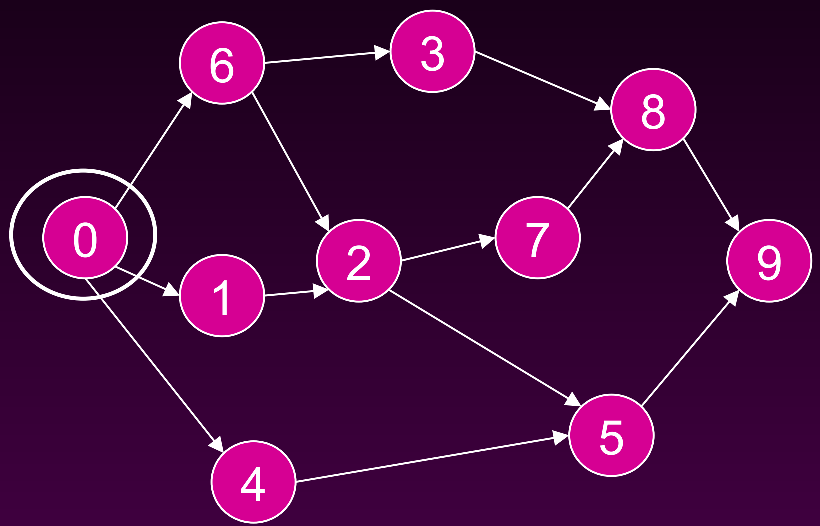
Indegree

0	0
1	1
2	2
3	1
4	1
5	2
6	1
7	1
8	2
9	2

start

Q = { 0 }

OUTPUT: **0**



0	6	1	4
1	2		
2	7	5	
3	8		
4	5		
5	9		
6	3	2	
7	8		
8	9		
9			

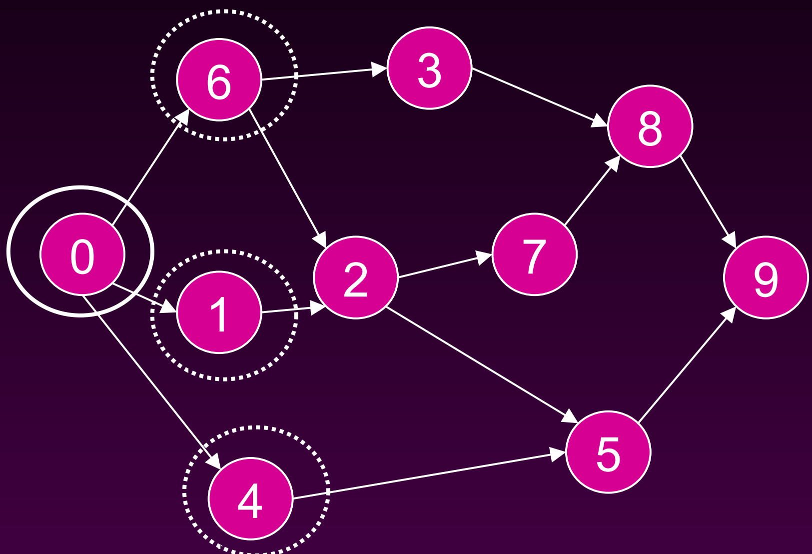
Indegree

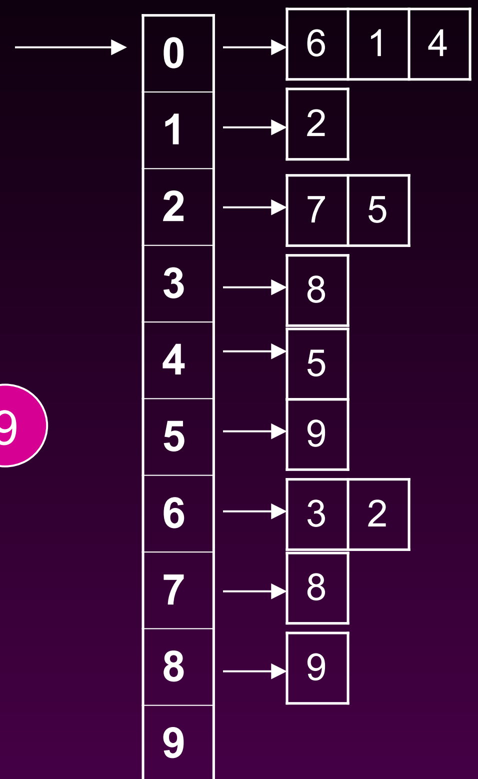
0	**0**	
1	**1**	-1
2	**2**	
3	**1**	
4	**1**	-1
5	**2**	
6	**1**	-1
7	**1**	
8	**2**	
9	**2**	

Dequeue 0 Q = { }
-> remove 0's arcs – adjust indegrees of neighbors

Decrement 0's neighbors

OUTPUT: **0**





Indegree

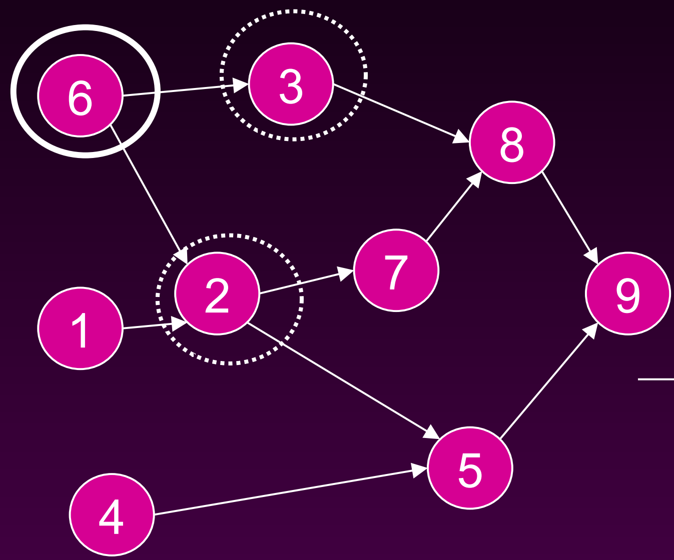
0	**0**
1	**0**
2	**2**
3	**1**
4	**0**
5	**2**
6	**0**
7	**1**
8	**2**
9	**2**

Q = { 6, 1, 4 }

Enqueue all starting points

Enqueue all new start points

OUTPUT: 0



0	6 1 4
1	2
2	7 5
3	8
4	5
5	9
6	3 2
7	8
8	9
9	

Indegree

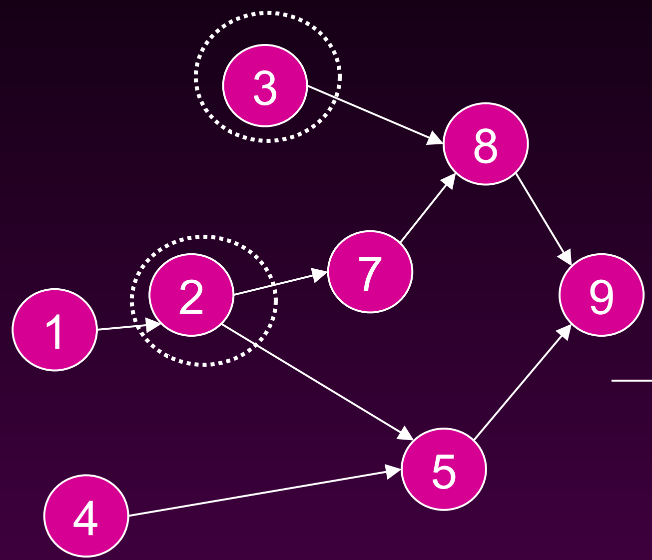
~~0~~	~~0~~	
~~1~~	~~0~~	
2	2	-1
3	1	-1
~~4~~	~~0~~	
5	2	
~~6~~	~~0~~	
7	1	
8	2	
9	2	

Dequeue 6 Q = { 1, 4 }

Remove arcs .. Adjust indegrees of neighbors

Adjust neighbors indegree

OUTPUT: 0 6



0	6, 1, 4
1	2
2	7, 5
3	8
4	5
5	9
6	3, 2
7	8
8	9
9	

Indegree

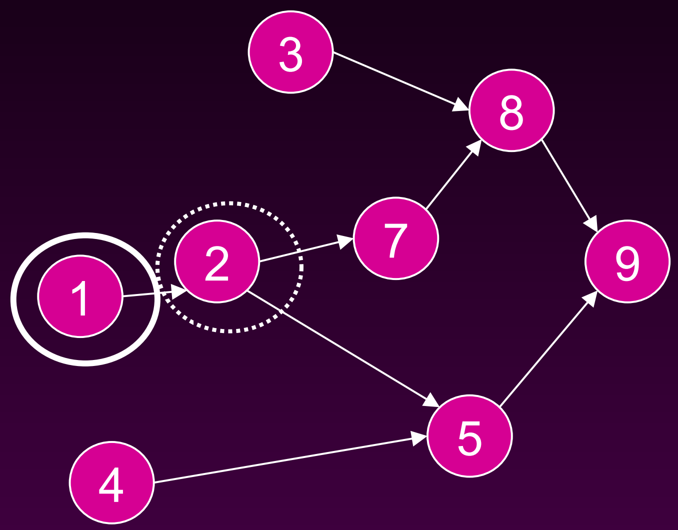
~~0~~	~~0~~
~~1~~	~~0~~
2	1
3	0
~~4~~	~~0~~
5	2
~~6~~	~~0~~
7	1
8	2
9	2

Q = { 1, 4, 3 }

Enqueue 3

Enqueue new start

OUTPUT: 0 6



0	6, 1, 4
1	2
2	7, 5
3	8
4	5
5	9
6	3, 2
7	8
8	9
9	

Indegree

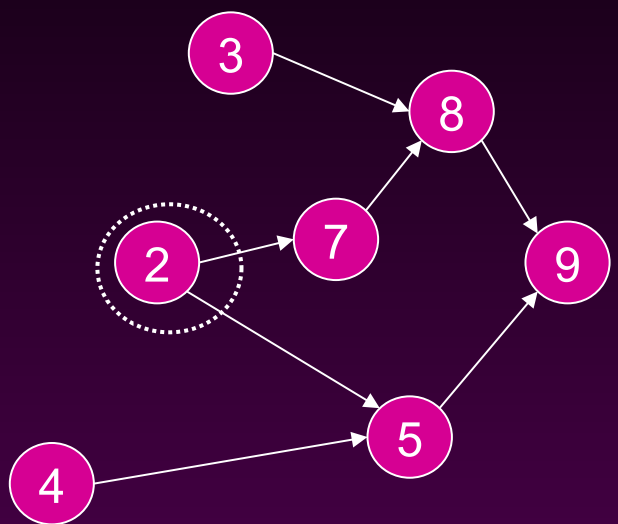
~~0~~	~~0~~	
~~1~~	~~0~~	
2	1	-1
~~3~~	~~0~~	
~~4~~	~~0~~	
5	2	
~~6~~	~~0~~	
7	1	
8	2	
9	2	

Dequeue 1 Q = { 4, 3 }

Adjust indegrees of neighbors

Adjust neighbors of 1

OUTPUT: 0 6 1



0	6	1	4
1	2		
2	7	5	
3	8		
4	5		
5	9		
6	3	2	
7	8		
8	9		
9			

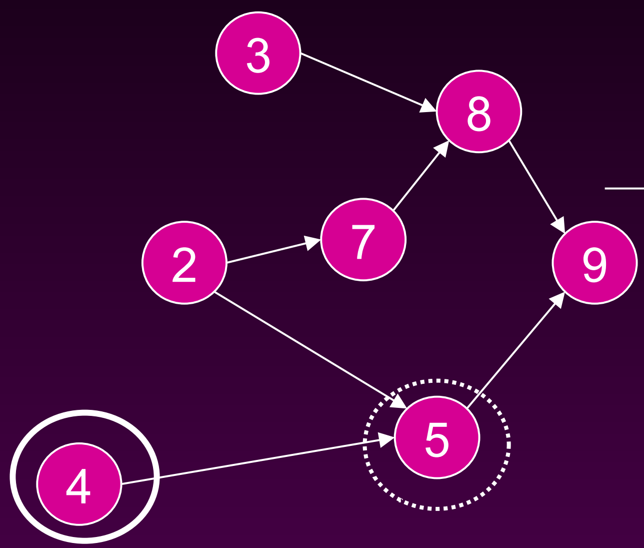
Indegree

~~**0**~~	~~**0**~~
~~**1**~~	~~**0**~~
2	**0**
~~**3**~~	~~**0**~~
~~**4**~~	~~**0**~~
5	**2**
~~**6**~~	~~**0**~~
7	**1**
8	**2**
9	**2**

Dequeue 1 Q = { 4, 3, 2 }
Enqueue 2

Enqueue new starting points

OUTPUT: 0 6 1



0	6, 1, 4
1	2
2	7, 5
3	8
4	5
5	9
6	3, 2
7	8
8	9
9	

Indegree

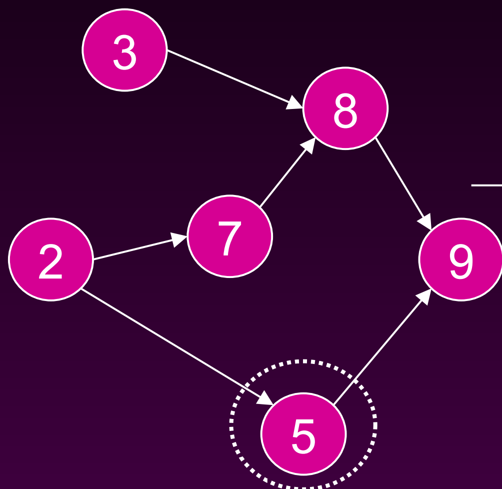
~~0~~	~~0~~	
~~1~~	~~0~~	
~~2~~	~~0~~	
~~3~~	~~0~~	
~~4~~	~~0~~	
5	2	-1
~~6~~	~~0~~	
7	1	
8	2	
9	2	

Dequeue 4 Q = { 3, 2 }

Adjust indegrees of neighbors

Adjust 4's neighbors

OUTPUT: 0 6 1 4



0	6 1 4
1	2
2	7 5
3	8
4	5
5	9
6	3 2
7	8
8	9
9	

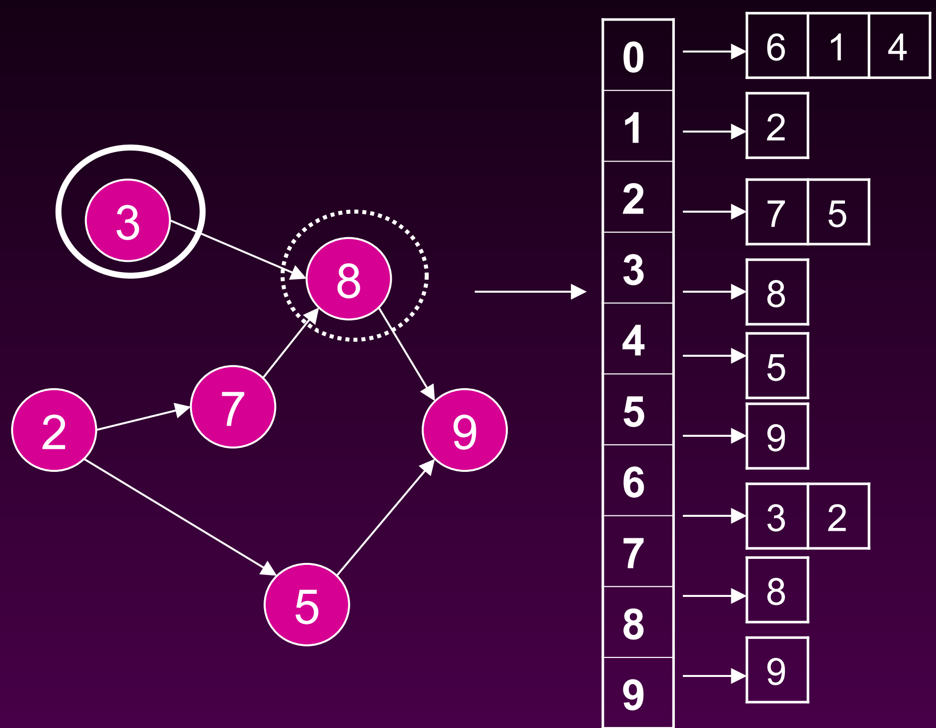
Indegree

~~0~~	~~0~~
~~1~~	~~0~~
~~2~~	~~0~~
~~3~~	~~0~~
~~4~~	~~0~~
5	1
~~6~~	~~0~~
7	1
8	2
9	2

Dequeue 4 Q = { 3, 2 }
No new start points found

NO new start points

OUTPUT: 0 6 1 4



Indegree

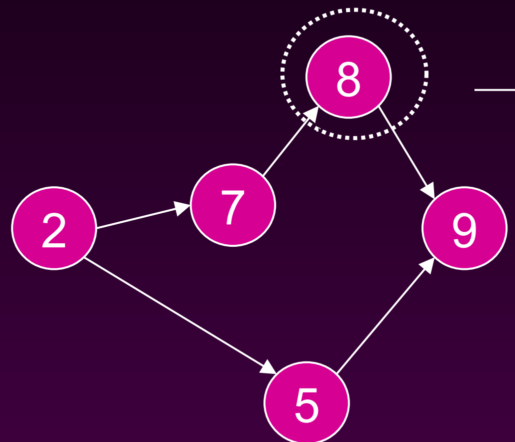
~~0~~	~~0~~
~~1~~	~~0~~
~~2~~	~~0~~
~~3~~	~~0~~
~~4~~	~~0~~
5	1
~~6~~	~~0~~
7	1
8	2
9	2

-1

Dequeue 3 Q = { 2 }

Adjust 3's neighbors

OUTPUT: 0 6 1 4 3



0	6	1	4
1	2		
2	7	5	
3	8		
4	5		
5	9		
6	3	2	
7	8		
8	9		
9			

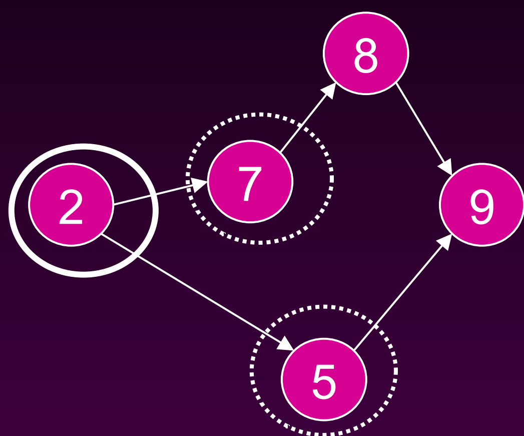
Indegree

~~0~~	~~0~~
~~1~~	~~0~~
~~2~~	~~0~~
~~3~~	~~0~~
~~4~~	~~0~~
5	1
~~6~~	~~0~~
7	1
8	1
9	2

Dequeue 3 Q = { 2 }

No new start points found

OUTPUT: 0 6 1 4 3



0	6, 1, 4
1	2
2	7, 5
3	8
4	5
5	9
6	3, 2
7	8
8	9
9	

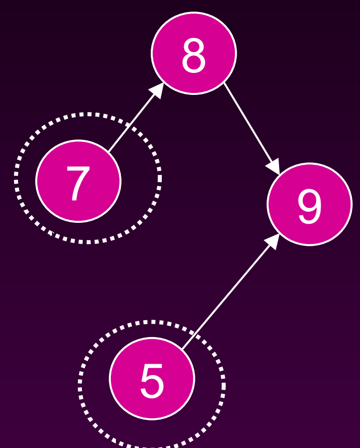
Indegree

~~0~~	~~0~~	
~~1~~	~~0~~	
~~2~~	~~0~~	
~~3~~	~~0~~	
~~4~~	~~0~~	
5	1	-1
~~6~~	~~0~~	
7	1	-1
8	1	
9	2	

Dequeue 2 Q = { }

Adjust 2’s neighbors

OUTPUT: 0 6 1 4 3 2



0	6	1	4
1	2		
2	7	5	
3	8		
4	5		
5	9		
6	3	2	
7	8		
8	9		
9			

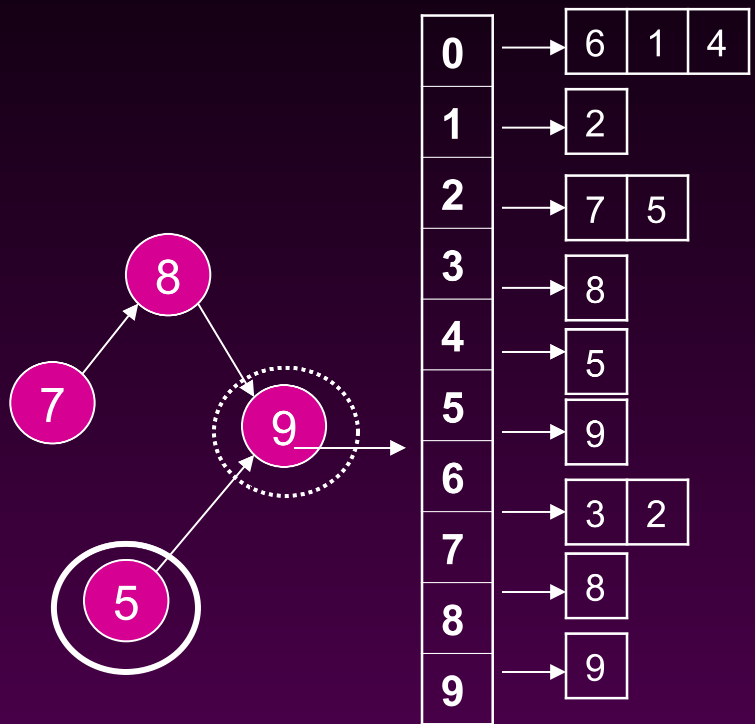
Indegree

0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	1
9	2

Dequeue 2 Q = { 5, 7 }

Enqueue 5, 7

OUTPUT: 0 6 1 4 3 2



Indegree

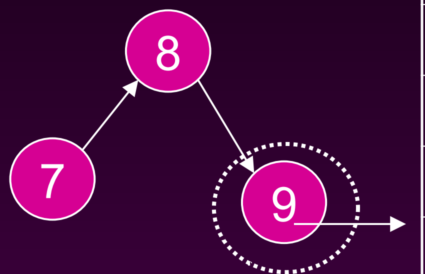
0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	1
9	2

-1

Dequeue 5 Q = { 7 }

Adjust neighbors

OUTPUT: 0 6 1 4 3 2 5



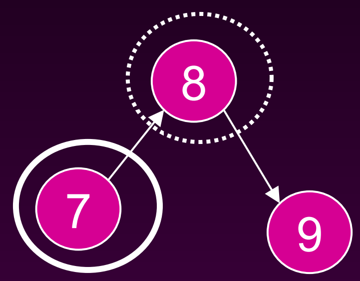
0	6, 1, 4
1	2
2	7, 5
3	8
4	5
5	9
6	3, 2
7	8
8	9
9	

Indegree

0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	1
9	1

Dequeue 5 Q = { 7 }
No new starts

OUTPUT: 0 6 1 4 3 2 5



0	6 1 4
1	2
2	7 5
3	8
4	5
5	9
6	3 2
7	8
8	9
9	

Indegree

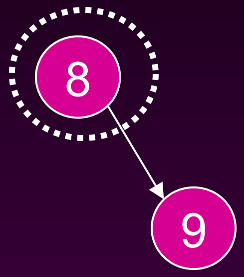
0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	1
9	1

-1

Dequeue 7 Q = { }

Adjust neighbors

OUTPUT: 0 6 1 4 3 2 5 7



0	6, 1, 4
1	2
2	7, 5
3	8
4	5
5	9
6	3, 2
7	8
8	9
9	

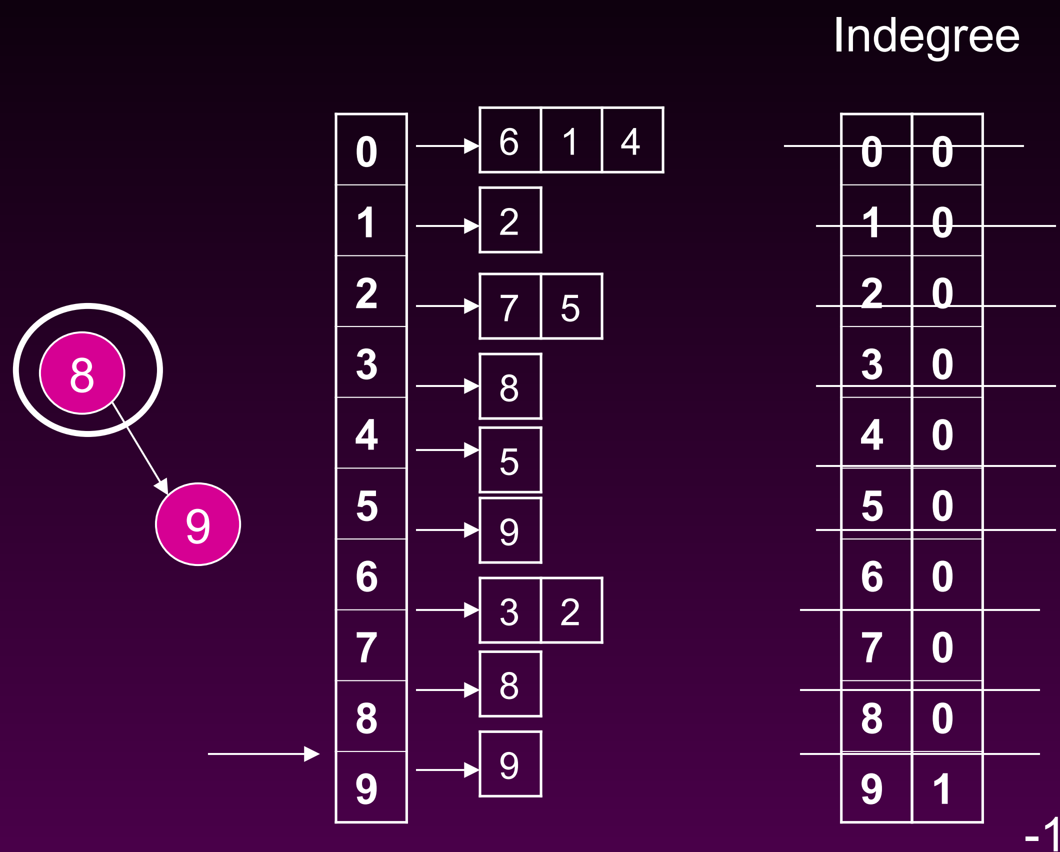
Indegree

0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	1

Dequeue 7 Q = { 8 }

Enqueue 8

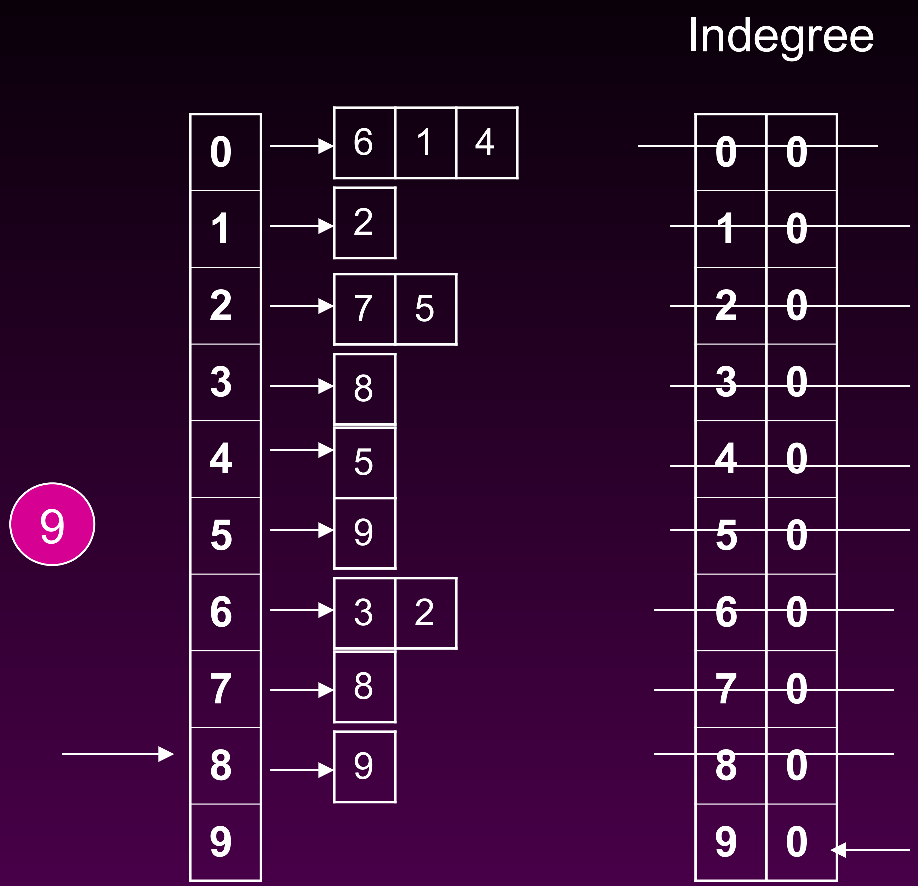
OUTPUT: 0 6 1 4 3 2 5 7



Dequeue 8 Q = { }

Adjust indegrees of neighbors

OUTPUT: 0 6 1 4 3 2 5 7 8



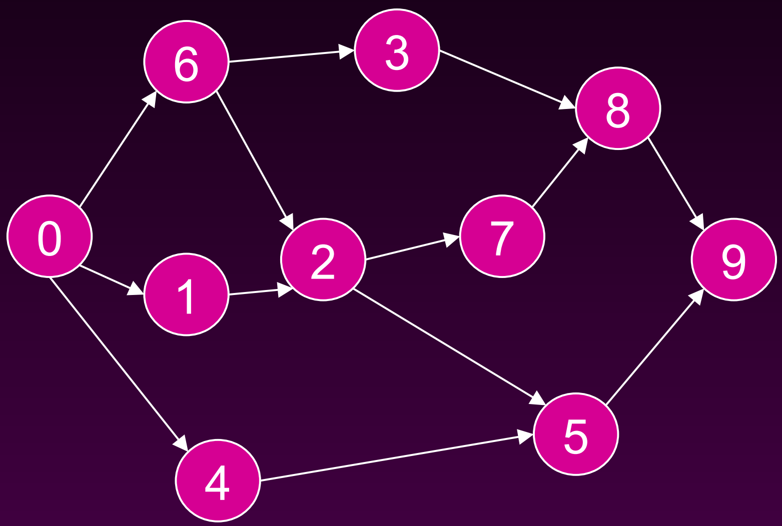
Dequeue 8 Q = { 9 }

Enqueue 9

Dequeue 9 Q = { }

STOP – no neighbors

OUTPUT: 0 6 1 4 3 2 5 7 8 9



OUTPUT: 0 6 1 4 3 2 5 7 8 9

Is output topologically correct?

Topological Sort: Complexity

- We never visited a vertex more than one time
- For each vertex, we had to examine all outgoing edges
  - Σ *outdegree(v) = m*
  - This is summed over all vertices, not per vertex
- So, our running time is exactly
  - O(n + m)

Summary:
Two representations:

- **Some definitions: …**
- **Two sizes: n = |V| and m=|E|,**
  - **m = O(n^2)**
- **Adjacency List**
  - More compact than adjacency matrices if graph has few edges
  - Requires a scan of adjacency list to check if an edge exists
  - Requires a scan to obtain all edges!
- **Adjacency Matrix**
  - Always require n^2 space
    - This can waste a lot of space if the number of edges are sparse
  - find if an edge exists in O(1)
  - Obtain all edges in O(n)
- O(n+m) for indegree for a DAG

(one), Two, (three) algorithms:

RDFS(v)

```
v is visited
W = {unvisited neighbors of v}
for each w in W
        RDFS(w)
```

BFS (queue)

```
s is visited
enqueue(Q,s)
while not-empty(Q)
        v <- dequeue(Q)
        W = {unvisited neighbors of v}
        for each w in W
                w is visited
                enqueue(Q,w)
```

DFS (stack)

```
s is visited
push(S,s)
while not-empty(S)
        v <- pop(S)
        W = {unvisited neighbors of v}
        for each w in W
                w is visited
                push(S,w)
```

Two applications

- For each non-visited vertex, run ‘connected component’ (either BFS or DFS)
  - For a connected component, list all vertices, find a spanning tree (BFS tree or DFS tree)
- ‘Shortest paths’ and ‘topological sort’ (for DAG only) are close to BFS