Uniform and Normal Distribution

1. Uniform or Rectangular Distribution

Let α and β be two real numbers such that $-\infty < \alpha < \beta < \infty$. A continuous random variable X is said to have a uniform (or rectangular) distribution over the interval (α, β) (written as $X \sim U(\alpha, \beta)$) if probability density function of X is given by

$$f_X(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } \alpha < x < \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Now, the r-th moment of $X \sim U(\alpha, \beta)$ is

$$\begin{aligned} E(X^r) &= \int_{-\infty}^{\infty} x^r f_X(x) dx \\ &= \int_{\alpha}^{\beta} \frac{x^r}{\beta - \alpha} dx \\ &= \frac{\beta^{r+1} - \alpha^{r+1}}{(r+1)(\beta - \alpha)} \\ &= \frac{\beta^r + \beta^{r-1}\alpha + \cdots + \beta\alpha^{r-1} + \alpha^r}{r+1}. \end{aligned}$$

Hence,

$$\begin{aligned} E(X) &= \frac{\alpha + \beta}{2}; \\ E(X^2) &= \frac{\beta^2 + \beta\alpha + \alpha^2}{3}; \\ Var(X) &= E(X^2) - (E(X))^2 = \frac{(\beta - \alpha)^2}{12}. \end{aligned}$$

The m.g.f. of $X \sim U(\alpha, \beta)$ is

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \\ &= \int_{\alpha}^{\beta} \frac{e^{tx}}{\beta - \alpha} dx \\ &= \begin{cases} \frac{e^{t\beta} - e^{t\alpha}}{(\beta-\alpha)t}, & \text{if } t \neq 0 \\ 1, & \text{if } t = 0. \end{cases} \end{aligned}$$

The d.f. of $X \sim U(\alpha, \beta)$ is

$$
\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(t)dt \\
&= \begin{cases} 0, \text{ if } x < \alpha \\ \frac{x-\alpha}{\beta-\alpha}, \text{ if } \alpha \le x < \beta \\ 1, \text{ if } x \ge \beta. \end{cases}
\end{aligned}
$$

Remark 1. *Let $X \sim U(\alpha, \beta)$ and $Y = \frac{X-\alpha}{\beta-\alpha}$. Then the d.f. of Y is*

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(X \le \alpha + (\beta - \alpha)y) \\
&= \begin{cases} 0, \ \textit{if } \alpha + (\beta - \alpha)y < \alpha \\ \frac{\alpha+(\beta-\alpha)y-\alpha}{\beta-\alpha}, \ \textit{if } \alpha \le \alpha + (\beta - \alpha)y < \beta \\ 1, \ \textit{if } \alpha + (\beta - \alpha)y \ge \beta \end{cases} \\
&= \begin{cases} 0, \ \textit{if } y < 0 \\ y, \ \textit{if } 0 \le y < 1 \\ 1, \ \textit{if } y \ge 1. \end{cases}
\end{aligned}
$$

Clearly, F_Y is not differentiable at 0 and 1. Hence, the p.d.f. of Y is

$$
f_Y(y) = \begin{cases} 1, \ \textit{if } 0 < y < 1 \\ 0, \ \textit{otherwise.} \end{cases}
$$

Therefore, $Y \sim U(0, 1)$.

Example 2. *Let $a > 0$ be a real constant. A point X is chosen at random on the interval $(0, a)$ (i.e., $X \sim U(0, a)$).*

(1) If Y denotes the area of equilateral triangle having sides of length X, find the mean and variance of Y.
(2) If the point X divides the interval $(0, a)$ into subintervals $I_1 = (0, X)$ and $I_2 = [X, a)$, find the probability that the larger of these two subintervals is at least the double of the size of the smaller subinterval.

Solution:

(1) We have $Y = \frac{\sqrt{3}}{4}X^2$. Then

$$
\begin{aligned}
E(Y) &= \frac{\sqrt{3}}{4}E(X^2) = \frac{\sqrt{3}}{12}a^2; \\
E(Y^2) &= \frac{3}{16}E(X^4) = \frac{3}{80}a^4; \\
Var(Y) = E(Y^2) - (E(Y))^2 &= \frac{a^4}{80}.
\end{aligned}
$$

(2) The required probability is

$$\begin{aligned}
p &= P(\{\max(X, a-X) \geq 2\min(X, a-X)\}) \\
&= P(\{a-X \geq 2X, X \leq \frac{a}{2}\}) + P(\{X \geq 2(a-X), X > \frac{a}{2}\}) \\
&= P(X \leq \frac{a}{3}\}) + P(\{X \geq \frac{2a}{3}\}) \\
&= F_X(\frac{a}{3}) + 1 - F_X(\frac{2a}{3}) \\
&= \frac{1}{3} + 1 - \frac{2}{3} = \frac{2}{3}.
\end{aligned}$$

2. Normal or Gaussian Distribution

(1) Let $\mu \in \mathbb{R}$ and $\sigma > 0$ be real constants. A continuous random variable X is said to have a normal (or Gaussian) distribution with parameters μ and σ^2 (written as $X \sim N(\mu, \sigma^2)$) if probability density function of X is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \ -\infty < x < \infty.$$

(2) The $N(0,1)$ distribution is called the standard normal distribution. The p.d.f. and the d.f. of $N(0,1)$ distributions will be denoted by ϕ and Φ respectively, i.e.,

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \ -\infty < z < \infty$$

$$\Phi(z) = \int_{-\infty}^{z} \phi(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx.$$

(3) We know that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ and $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$.

Clearly if $X \sim N(\mu, \sigma^2)$, then

$$f_X(\mu - x) = f_X(\mu + x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \ \forall\, x \in \mathbb{R}.$$

Thus, the distribution of X is symmetric about μ. Hence,

$$X \sim N(\mu, \sigma^2) \Rightarrow F_X(\mu - x) + F_X(\mu + x) = 1, \ \forall\, x \in \mathbb{R} \text{ and } F_X(\mu) = \frac{1}{2}.$$

In particular,

$$\boxed{\Phi(-z) = 1 - \Phi(z), \ \forall\, z \in \mathbb{R} \text{ and } \Phi(0) = \frac{1}{2}.}$$

Suppose that $X \sim N(\mu, \sigma^2)$. Then the p.d.f. of $Z = \frac{X-\mu}{\sigma}$ is given by

$$\begin{aligned}
f_Z(z) &= f_X(\mu + \sigma z)|\sigma|, \ -\infty < z < \infty \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \ -\infty < z < \infty
\end{aligned}$$

i.e.,

$$\boxed{X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{X-\mu}{\sigma} \sim N(0,1).}$$

Thus

$$\boxed{X \sim N(\mu, \sigma^2) \Rightarrow F_X(x) = P(\{X \leq x\}) = P\left(\left\{Z \leq \frac{x-\mu}{\sigma}\right\}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right), \ \forall\, x \in \mathbb{R}.}$$

Now, the m.g.f. of $X \sim N(\mu, \sigma^2)$ is

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \int\limits_{-\infty}^{\infty} e^{tx} f_X(x) dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{e^{\mu t}}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} e^{-y^2+\sqrt{2}\sigma t y} dy \text{ (by putting } \frac{x-\mu}{\sqrt{2}\sigma} = y) \\ &= \frac{e^{(\mu t + \frac{\sigma^2 t^2}{2})}}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} e^{-(y-\frac{\sqrt{2}\sigma t}{2})^2} dy \\ &= e^{(\mu t + \frac{\sigma^2 t^2}{2})}, \ \forall \ t \in \mathbb{R}. \end{aligned}$$

Therefore,

$$\begin{aligned} &M_X^{(1)}(t) = (\mu + \sigma^2 t) e^{(\mu t + \frac{\sigma^2 t^2}{2})}, \ \forall \ t \in \mathbb{R}; \\ &M_X^{(2)}(t) = (\sigma^2 + (\mu + \sigma^2 t)^2) e^{(\mu t + \frac{\sigma^2 t^2}{2})}, \ \forall \ t \in \mathbb{R}; \\ &E(X) = M_X^{(1)}(0) = \mu; \\ &E(X^2) = M_X^{(2)}(0) = \mu^2 + \sigma^2; \\ &\text{and } Var(X) = E(X^2) - (E(X))^2 = \sigma^2. \end{aligned}$$