Types of Random Vector

Let $(\mathcal{S}, \Sigma, P)$ be a probability space and let $\underline{X} = (X, Y) : \mathcal{S} \longrightarrow \mathbb{R}^2$ be a random vector with joint distribution function $F_{\underline{X}}$.

Notations.

- Let $\mathbb{B}_{\mathbb{R}^n}$ denotes the set which contains all rectangles (Cartesian product of open, closed and semi-closed intervals) and their countable union and intersection.
- Let I_n be a rectangle in $\mathbb{R}^n$. We will denote by $\mathbb{B}_{I_n}$ the set which contains all rectangles contained in I_n and their countable union and intersection.

Definition 1. *$\underline{X}$ is a said to be a random vector of discrete type if there exists a non-empty countable set $E_{\underline{X}} \subset \mathbb{R}^2$ such that $P(\underline{X} = \underline{x}) > 0$, for every $\underline{x} \in E_{\underline{X}}$, and $P(\underline{X} \in E_{\underline{X}}) = 1$.*

The set $E_{\underline{X}}$ is called the support of $\underline{X}$.

The function $f_{\underline{X}} : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f_{\underline{X}}(\underline{x}) = P(\underline{X} = \underline{x}) = P(X = x, Y = y)$$

is called the joint probability mass function of $\underline{X}$.

Remark 2. *Let $\underline{X}$ be a random vector of discrete type with support $E_{\underline{X}}$, joint d.f. $F_{\underline{X}}$ and joint p.m.f. $f_{\underline{X}}$.*

(1) $\sum_{\underline{x} \in E_{\underline{X}}} f_{\underline{X}}(\underline{x}) = 1$. *Moreover, $P(\underline{X} \in E^c_{\underline{X}}) = 0$ and $f_{\underline{X}}(\underline{x}) = 0$, $\forall\ \underline{x} \in E^c_{\underline{X}}$.*
(2) *For any $A \in \mathbb{B}_{\mathbb{R}^2}$,*

$$P(\underline{X} \in A) = \sum_{\underline{x} \in A \cap E_{\underline{X}}} f_{\underline{X}}(\underline{x}) = \sum_{\underline{x} \in E_{\underline{X}}} f_{\underline{X}}(\underline{x}) I_A(\underline{x}).$$

(3) *For $\underline{x} \in \mathbb{R}^2$,*

$$F_{\underline{X}}(\underline{x}) = P(\underline{X} \in (-\underline{\infty}, \underline{x}]) = \sum_{\underline{x} \in (-\underline{\infty}, \underline{x}] \cap E_{\underline{X}}} f_{\underline{X}}(\underline{x}).$$

Definition 3. *$\underline{X}$ is a said to be a random vector of continuous type if there exists a nonnegative function $f_{\underline{X}} : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$F_{\underline{X}}(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{\underline{X}}(x, y) dx dy.$$

The set $E_{\underline{X}} = \{\underline{x} \in \mathbb{R}^2 : f_{\underline{X}}(\underline{x}) > 0\}$ is called the support of $\underline{X}$.

The function $f_{\underline{X}}$ is called the joint probability density function of $\underline{X}$.

Remark 4. *Let $\underline{X}$ be a random vector of continuous type with support $E_{\underline{X}}$, joint d.f. $F_{\underline{X}}$ and joint p.d.f. $f_{\underline{X}}$.*

(1) *For any $\underline{x} \in \mathbb{R}^2$, $f_{\underline{X}}(\underline{x}) \geq 0$, and*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\underline{X}}(x, y) dx dy = 1.$$

(2) *For any $\underline{x} \in \mathbb{R}^2$, $P(\underline{X} = \underline{x}) = 0$. Consequently, for any countable set $S \subset \mathbb{R}^2$, $P(\underline{X} \in S) = 0$.*

(3) *Let* $\underline{a} = (a_1, a_2)$, $\underline{b} = (b_1, b_2) \in \mathbb{R}^2$ *such that* $a_i < b_i$, $i = 1, 2$. *Let* $(\underline{a}, \underline{b}] = (a_1, a_2] \times (b_1, b_2]$. *Then*

$$P(\underline{X} \in (\underline{a}, \underline{b}]) = P(a_1 < X \leq b_1, a_2 < Y \leq b_2) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f_{\underline{X}}(x, y)dxdy.$$

Theorem 5. *Let* $\underline{X} = (X, Y) : \mathcal{S} \longrightarrow \mathbb{R}^2$ *be a random vector with joint distribution function* $F_{\underline{X}}$.

(1) *Suppose that* $\underline{X}$ *is of discrete type with support* $E_{\underline{X}}$ *and joint p.m.f.* $f_{\underline{X}}$. *Define*

$$R_x = \{y \in \mathbb{R} : (x, y) \in E_{\underline{X}}\}, \ R_y = \{x \in \mathbb{R} : (x, y) \in E_{\underline{X}}\}.$$

Then X *and* Y *are of discrete type with support*

$$E_X = \{x \in \mathbb{R} : (x, y) \in E_{\underline{X}} \text{ for some } y \in \mathbb{R}\}$$

and

$$E_Y = \{y \in \mathbb{R} : (x, y) \in E_{\underline{X}} \text{ for some } x \in \mathbb{R}\}$$

respectively. The marginal p.m.f.s of X *and* Y *are respectively given by*

$$f_X(x) = \begin{cases} \sum_{y \in R_x} f_{\underline{X}}(x, y), & \text{if } x \in E_X, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$f_Y(y) = \begin{cases} \sum_{x \in R_y} f_{\underline{X}}(x, y), & \text{if } y \in E_Y, \\ 0, & \text{otherwise}. \end{cases}$$

(2) *Suppose that* $\underline{X}$ *is of continuous type with support* $E_{\underline{X}}$ *and joint p.d.f.* $f_{\underline{X}}$. *Then* X *and* Y *are of continuous type with marginal p.d.f.s given by*

$$f_X(x) = \int_{-\infty}^{\infty} f_{\underline{X}}(x, y)dy \text{ and } f_Y(y) = \int_{-\infty}^{\infty} f_{\underline{X}}(x, y)dx$$

respectively.

Example 6. *Let* $\underline{X} = (X, Y)$ *be a random vector with joint p.m.f.*

$$f_{\underline{X}}(x, y) = \begin{cases} cy, & \text{if } (x, y) \in A, \\ 0, & \text{otherwise}; \end{cases}$$

where $A = \{(a, b) : a, b \in \{1, 2, \ldots, n\}, \ a \leq b\}$, $n \geq 2$ *is a fixed integer and* c *is a constant.*

(1) *Find the value of* c.
(2) *Find the marginal p.m.f.s of* X *and* Y.
(3) *Find* $P(X > Y)$, $P(X = Y)$ *and* $P(X < Y)$.

Solution.

(1) Clearly $c > 0$. The support $E_{\underline{X}}$ of $\underline{X}$ is A. Therefore, $\sum_{(x,y) \in E_{\underline{X}}} f_{\underline{X}}(x, y) = 1$. This implies that $c \sum_{y=1}^{n} \sum_{x=1}^{y} y = 1$ or $c \sum_{y=1}^{n} y^2 = 1$. Thus, $c = \frac{6}{n(n+1)(2n+1)}$.
(2) The support of X is $E_X = \{1, 2, \ldots, n\}$ and the support of Y is $E_Y = \{1, 2, \ldots, n\}$. For $x \in E_X$, we have $R_x = \{x, x+1, \ldots, n\}$ and

$$\sum_{y \in R_x} f_{\underline{X}}(x, y) = c \sum_{y=x}^{n} y = c \left[\frac{n(n+1)}{2} - \frac{(x-1)x}{2} \right].$$

The marginal p.m.f. of X is then

$$f_X(x) = \begin{cases} \frac{3[n(n+1)-(x-1)x]}{n(n+1)(2n+1)}, & \text{if } x \in E_X, \\ 0, & \text{otherise.} \end{cases}$$

For $y \in E_Y$, we have $R_y = \{1, 2, \ldots, y\}$ and

$$\sum_{x \in R_y} f_{\underline{X}}(x, y) = c \sum_{x=1}^{y} y = cy^2.$$

The marginal p.m.f. of Y is then

$$f_Y(y) = \begin{cases} \frac{3y^2}{n(n+1)(2n+1)}, & \text{if } x \in E_Y, \\ 0, & \text{otherise.} \end{cases}$$

(3) Let $A = \{(a, b) : a > b\}$ and $B = \{(a, b) : a = b\}$. Then

$$\begin{aligned} P(X > Y) &= P(\underline{X} \in A) \\ &= \sum_{(x,y) \in E_{\underline{X}} \cap A} f_{\underline{X}}(x, y) \\ &= 0. \end{aligned}$$

$$\begin{aligned} P(X = Y) &= P(\underline{X} \in B) \\ &= \sum_{(x,y) \in E_{\underline{X}} \cap B} f_{\underline{X}}(x, y) \\ &= c \sum_{y=1}^{n} y = \frac{3}{2n+1}. \end{aligned}$$

Therefore, $P(X < Y) = \frac{2(n-1)}{2n+1}$.

Example 7. *Let $\underline{X} = (X, Y)$ be a random vector with joint p.d.f.*

$$f_{\underline{X}}(x, y) = \begin{cases} \frac{c}{x}, & \text{if } 0 < y < x < 1, \ c \in \mathbb{R}, \\ 0, & \text{otherwise.} \end{cases}$$

(1) *Find the value of c.*
(2) *Find the marginal p.d.f.s of X and Y.*
(3) *Find $P(X > 2Y)$.*

Solution.

(1) Since $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\underline{X}}(x, y) dx dy = 1$. This implies that $c \int_0^1 \int_0^x \frac{1}{x} dy dx = 1$ or $c \int_0^1 dx = 1$ or $c = 1$.
(2) The marginal p.d.f. of X is given by

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{\underline{X}}(x, y) dy \\ &= \begin{cases} \int_0^x \frac{1}{x} dy, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The marginal p.d.f. of Y is given by

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{\underline{X}}(x,y)dx \\ &= \begin{cases} \int_y^1 \frac{1}{x}dx, & \text{if } 0 < y < 1, \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} -\ln y, & \text{if } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(3) Let $A = \{(x,y) : x > 2y\}$. Then

$$\begin{aligned} P(X > 2Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\underline{X}}(x,y) I_A(x,y)dydx \\ &= \iint\limits_{0<2y<x<1} \frac{1}{x}dydx \\ &= \int_0^1 \int_0^{x/2} \frac{1}{x}dydx \\ &= \frac{1}{2}. \end{aligned}$$