Probability and Statistics
Problem Set IV

Note: For $n \geq 2$, let $\underline{X} = (X_1, X_2, \cdots, X_n)$ be a random vector with joint cumulative distribution function F. We can also say that $(X_1, X_2, \cdots, X_n)$ is an n-dimensional random variable with joint cumulative distribution function F or $X_1, X_2, \cdots, X_n$ are random variables with joint cumulative distribution function F.

1. Check whether the following functions are joint cdf of some random vector or not.

   (a)
$$F(x, y) = \begin{cases} 1 & \text{if } x + 2y \geq 1 \\ 0 & \text{if } x + 2y < 1 \end{cases}$$

   (b)
$$F(x, y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x + y < 1 \text{ or } y < 0 \\ 1 & \text{otherwise} \end{cases}$$

   Answer: (a) No, (b) No

2. Suppose a fair coin is tossed three times. Let X be the number of heads and Y be the absolute difference between number of heads and number of tails in three tossing. Then show that $\underline{Z} = (X, Y)$ is a discrete type random vector.

3. Let S be a sample space and P be a probability function defined for all events. Let A and B be two events. Define the following random variables
$$X(w) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{otherwise} \end{cases} \text{ and } Y(w) = \begin{cases} 1 & \text{if } w \in B \\ 0 & \text{otherwise} \end{cases}.$$
   Then show that X and Y are independent if and only if A and B are independent.

4. Let X_1 and X_2 be independent identical distributed random variables with common p.m.f. $P(X = \pm 1) = \frac{1}{2}$. Show that X_1, X_2, X_3 are pairwise independent but not independent, where $X_3 = X_1 X_2$.

5. Let X and Y be two random variables with joint p.d.f.
$$f(x, y) = \begin{cases} \frac{1+xy}{4} & \text{if } |x| < 1, |y| < 1 \\ 0 & \text{otherwise} \end{cases}$$
   Then show that X and Y are not independent but X^2 and Y^2 are independent.

6. Toss a fair coin 3 times. Let X be the number of heads on the first toss, Y be the total number of heads on the last two tosses, and Z be the number of heads on the first two tosses.

   (a) Find the joint p.m.f. for X and Y. Compute $\text{Cov}(X, Y)$.
   (b) Find the joint p.m.f. for X and Z. Compute $\text{Cov}(X, Z)$.

   Answer: Calculate the joint p.m.f.s yourselves. Show that X and Y are independent.
   (b) $\text{Cov}(X, Z) = 1/4$

7. Roll a dice ($n = 1, 2, ...6$). Two events s_1 and s_2 are defined as follows:

$$s_1 = \begin{cases} 1 & \text{if } n \text{ is even number} \\ 0 & \text{otherwise} \end{cases}$$

$$s_2 = \begin{cases} 1 & \text{if n is prime number} \\ 0 & \text{otherwise} \end{cases}$$

Find the joint p.m.f. $P(s_1, s_2)$. Also find the covariance and correlation coefficient between s_1 and s_2.
Answer: $P(1,1) = P(0,0) = 1/6, P(1,0) = P(0,1) = 2/6$, $\text{Cov}(s_1, s_2) = -\frac{1}{12}$ $\rho(s_1, s_2) = -\frac{1}{3}$

8. The joint probability mass function of two discrete random variables X and Y is given by

$$p(x,y) = \begin{cases} c(2x+y) & \text{if } (x,y) \in \{0,1,2\} \times \{0,1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the value of c.
   (b) Find $P(X = 2, Y = 1)$, $P(X \geq 1, Y \leq 2)$
   (c) Find the marginal p.m.f. of X and Y.
   (d) Are X and Y independent.

Answer:(a) $\frac{1}{42}$, (b) $\frac{5}{42}$ & $\frac{4}{7}$(c)The p.m.f. of X is

$$p_X(x) = \begin{cases} \frac{1}{7} & \text{if } x = 0 \\ \frac{1}{3} & \text{if } x = 1 \\ \frac{11}{21} & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

and the p.m.f. of Y is

$$p_Y(y) = \begin{cases} \frac{1}{7} & \text{if } y = 0 \\ \frac{3}{14} & \text{if } y = 1 \\ \frac{2}{7} & \text{if } y = 2 \\ \frac{5}{14} & \text{if } y = 3 \\ 0 & \text{otherwise} \end{cases}$$

(d) No

9. **(a)** Prove that the correlation $\rho(X,Y)$ between two random variables X and Y remains unchanged under the transformation $X = aX + b$, where a, b are constants.

   (b) Recall the relation between degrees Fahrenheit and degrees Celsius

$$\text{degrees Celsius} = \frac{5}{9} \text{ degrees Fahrenheit } - \frac{160}{9}$$

Let X and Y be the daily high temperature in degrees Fahrenheit for the summer in Delhi and Allahabad. Let T and S be the same temperatures in degrees Celsius.

Suppose that $\text{Cov}(X, Y) = 4$ and $\rho(X, Y) = 0.8$. Compute $\text{Cov}(T, S)$ and $\rho(T, S)$.
Answer: $\text{Cov}(T, S) = 100/81$, $\rho(T, S) = 0.8$

10. A card is drawn at random from a normal 52-card pack and its identity noted. The card is replaced, the pack shuffled and the process repeated. Random variables W, X, Y , Z are defined as follows:

    W = 2 if the drawn card is a heart; W = 0 otherwise.

    X = 4 if the drawn card is an ace, king, or queen; X = 2 if the card is a jack or ten; X = 0 otherwise.

    Y = 1 if the drawn card is red; Y = 0 otherwise.

    Z = 2 if the drawn card is black and an ace, king or queen; Z = 0 otherwise.

    Establish the correlations between each pair of random variables W, X, Y , Z.
    Answer: $\rho(X, Z) = 0.598, \rho(Y, Z) = -0.361, \rho(W, Z) = -0.209, \rho(Y, W) = 0.577$.

11. The probability distribution for the number of eggs in a clutch is $P(\lambda)$, and the probability that each egg will hatch is p (independently of the size of the clutch). Show by direct calculation that the probability distribution for the number of chicks that hatch is $P(\lambda p)$. ($P(\lambda)$ denotes Poisson distribution with mean λ).

12. A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him to safety after two hours of travel. The second door leads to a tunnel that returns him to the mine after three hours of travel. The third door leads to a tunnel that returns him to his mine after five hours. Assuming that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until the miner reaches safety?
    Answer: 10.

13. A and B roll a pair of dice in turn, with A rolling first. A's objective is to obtain a sum of 6, and B's is to obtain a sum of 7. The game ends when either player reaches his or her objective, and that player is declared the winner. (a) Find the probability that A is the winner. (b) Find the expected number of rolls of the dice. (c) Find the variance of the number of rolls of the dice.
    Answer: (a) 30/61 (b) 402/61, (c) Var(X)$\approx$ 36.24

14. At a party N men throw their hats into the center of a room. The hats are mixed up and each man randomly selects one. Let X be the number of men those get their own hat. Find $E(X)$ and $Var(X)$.
    Answer: 1 and 1

15. Let (X, Y) have the joint p.m.f as follows

$$p(x,y) = \begin{cases} \frac{2}{15} & \text{if } (x,y) = (1,1) \\ \frac{4}{15} & \text{if } (x,y) = (1,2) \\ \frac{3}{15} & \text{if } (x,y) = (1,3) \\ \frac{1}{15} & \text{if } (x,y) = (2,1) \\ \frac{1}{15} & \text{if } (x,y) = (2,2) \\ \frac{4}{15} & \text{if } (x,y) = (2,3) \\ 0 & \text{otherwise} \end{cases}$$

Find the correlation coefficient of X and Y.
Answer: $\frac{21}{\sqrt{7236}}$

16. Let $X_1, X_2, \cdots, X_n$ be independent random variables. Suppose that $X_i's$ are continuous type with same cumulative distribution function for every $1 \leq i \leq n$. Let $Y_1 = max(X_1, X_2, \cdots, X_n)$ and $Y_2 = min(X_1, X_2, \cdots, X_n)$. Find the cumulative distribution functions of Y_1 and Y_2.
Answer: Suppose F is the c.d.f. of $X_i's$. Then the c.d.f. of Y_1 is $(F)^n$ and the c.d.f. of Y_2 is $1 - (1 - F)^n$.

17. If the number of typographical errors per page type by a certain typist follows a Poisson distribution with a mean of λ, find the probability that the total number of errors in 10 randomly selected pages is 10.
Answer: $\frac{e^{-10\lambda}(10\lambda)^{10}}{10!}$

18. We start with a stick of length l. We break it at a point which is chosen randomly and uniformly over its length and keep the piece that contains the left end of the stick. We then repeat the same process on the piece that we were left with. What is the expected length of the piece that we are left with after breaking twice?
Answer: $\frac{l}{4}$

19. Let $X_1, X_2, \cdots, X_n$ be independent random variable such that $X_i \sim N(\mu_i, \sigma_i^2)$, $\mu_i \in \mathbb{R}, \sigma_i > 0$, $i = 1, \cdots, n$. If $a_1, a_2, \cdots, a_n$ are real constant, such that not all of them are zero, then show that $\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$

20. Let $X_1, X_2, \cdots, X_n$ be independent random variable such that $X_i \sim \text{NB}(r_i, p)$, $r_i \in \mathbb{N}$, $i = 1, \cdots, n$ and $0 < p < 1$. Then show that $\sum_{i=1}^{n} X_i \sim \text{NB}\left(\sum_{i=1}^{n} r_i, p\right)$

21. Let (X, Y) be random vector with joint p.d.f.

$$f(x,y) = \begin{cases} cx + 1 & \text{if } x, y \geq 0 \text{ and } x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c.

(b) Find the marginal p.d.f. of X and Y.

(c) Find $P(Y < 2X^2)$.

Answer:(a) 3, (b)The p.d.f. of X is

$$f_X(x) = \begin{cases} (3x+1)(1-x) & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and the p.d.f. of Y is

$$f_Y(y) = \begin{cases} \frac{1}{2}(5-3y)(1-y) & \text{if } 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) $\frac{53}{96}$

22. Let (X, Y) be random vector with joint p.d.f.

$$f(x, y) = \begin{cases} 6e^{-(2x+3y)} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent.

(b) Find $E(Y|X > 2)$.

(c) Find $P(X > Y)$.

Answer:(a)Yes, (b)$\frac{1}{3}$, (c)$\frac{3}{5}$.

23. Let X be random variable with p.d.f.

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

We know that given $X = x$, the random variable Y uniformly distributed on $[-x, x]$.

(a) Find joint p.d.f. of X and Y.

(b) Find p.d.f. of Y.

(c) Find $P(|Y| < X^3)$.

Answer:(a)The joint p.d.f. is

$$f(x, y) = \begin{cases} 1 & \text{if } |y| \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) the p.d.f. of Y is

$$f_Y(y) = \begin{cases} 1-|y| & \text{if } |y| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) $\frac{1}{2}$

24. Let X be random variable with p.d.f.

$$f_X(x) = \begin{cases} 4x(1-x^2) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

For a fixed $x \in (0, 1)$, the conditional p.d.f. of Y given $X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{1-x^2} & \text{if } x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the conditional p.d.f. of X given $Y = y$ for appropriate values of y

(b) Find $E(X|Y = 0.5)$ and $\text{Var}(X|Y = 0.5)$.

(c) Find $P(0 < Y < \frac{1}{3})$ and $P(\frac{1}{3} < Y < \frac{2}{3}|X = 0.5)$.

Answer: (a) For $0 < y < 1$

$$f_{X|Y}(x|y) = \begin{cases} \frac{2x}{y^2} & \text{if } 0 < x < y \\ 0 & \text{otherwise} \end{cases}$$

(b)$E(X|Y = 0.5) = \frac{1}{3}$ and $\text{Var}(X|Y = 0.5) = \frac{1}{72}$, (c)$P(0 < Y < \frac{1}{3}) = \frac{1}{81}$ and $P(\frac{1}{3} < Y < \frac{2}{3}|X = 0.5) = \frac{7}{27}$.

25. Let (X, Y) be random vector with joint p.d.f.

$$f(x, y) = \begin{cases} \frac{1}{2}ye^{-xy} & \text{if } 0 < x < \infty, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(e^{\frac{X}{2}}|Y = 1)$.
Answer: 2

26. Let $\underline{X} = (X_1, X_2)$ be a random vector with joint p.d.f.

$$f(x_1, x_2) = \begin{cases} 2 & \text{if } 0 \le x_1 \le x_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the cumulative distribution function of $Y = X_1 + X_2$ and hence find the p.d.f. of Y.
Answer:(a)The c.d.f. is

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{y^2}{2} & \text{if } 0 \le y < 1 \\ 1 - \frac{(y-2)^2}{2} & \text{if } 1 \le y < 2 \\ 1 & \text{if } y \ge 2 \end{cases}$$

(b) the p.d.f. of Y is

$$f_Y(y) = \begin{cases} y & \text{if } 0 \le y \le 1 \\ 2 - y & \text{if } 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

27. Let $\underline{X} = (X_1, X_2)$ be a random vector with joint p.d.f.

$$f(x_1, x_2) = \begin{cases} 8x_1x_2 & \text{if } 0 < x_1 < x_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the pdf of $Y_1 = \frac{X_1}{X_2}$.
Answer: The p.d.f. is

$$f_{Y_1}(y) = \begin{cases} 2y & \text{if } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

28. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} \frac{e^{-(y+\frac{x}{y})}}{y} & \text{if } 0 < x, y < \infty \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that f is a joint probability density function of some random vector (X, Y).

(b) Find $\text{Cov}(X, Y)$.

Answer: 1

29. Let X and Y be jointly continuous random variables with joint p.d.f.

$$f(x,y) = \begin{cases} x + cy^2 & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the constant c.

(b) Find joint c.d.f.

(c) $P(0 \le X \le \frac{1}{2}, 0 \le Y \le \frac{1}{2})$.

(d) Find the marginal p.d.f. of X and Y.

Answer: (a) $\frac{3}{2}$, (b)The joint c.d.f. is

$$F_{X,Y}(x,y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ \frac{1}{2}x^2y + \frac{1}{2}xy^3 & \text{if } 0 \le x, y \le 1 \\ \frac{1}{2}x^2 + \frac{1}{2}x & \text{if } 0 \le x \le 1 \text{ and } y \ge 1 \\ \frac{1}{2}y + \frac{1}{2}y^3 & \text{if } 0 \le y \le 1 \text{ and } x \ge 1 \\ 1 & \text{if } x \ge 1 \text{ and } y \ge 1 \end{cases}$$

, (c) $\frac{3}{32}$, (d)The p.d.f. of X is

$$f_X(x) = \begin{cases} x + \frac{1}{2} & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and the p.d.f. of Y is

$$f_Y(y) = \begin{cases} \frac{3}{2}y^2 + \frac{1}{2} & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

30. Let $\underline{X} = (X_1,\, X_2)$ be a random vector with joint probability density function

$$f_X(x_1,x_2) = \begin{cases} \frac{1}{2}e^{-x_1} & \text{if } 0 < x_2 < x_1 < \infty \\ \frac{1}{2}e^{-x_2} & \text{if } 0 < x_1 < x_2 < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find the joint probability density function of $Y_1 = X_1 + X_2$ and $Y_2 = \dfrac{X_2}{X_1 + X_2}$.

Answer:

$$f_{(Y_1,\,Y_2)}(y_1,y_2) = \begin{cases} \dfrac{1}{2}|y_1|e^{-y_1(1-y_2)} & \text{if } 0 < y_1y_2 < y_1(1-y_2) < \infty \\ \dfrac{1}{2}|y_1|e^{-y_1y_2} & \text{if } 0 < y_1(1-y_2) < y_1y_2 < \infty \\ 0 & \text{otherwise} \end{cases}$$

31. Let X_1, X_2, X_3 be independent identically distributed $Exp(1)$ random variables. Find the joint p.d.f. of $Y_1 = X_1 + X_2 + X_3$, $Y_2 = \frac{X_1+X_2}{X_1+X_2+X_3}$, and $Y_3 = \frac{X_1}{X_1+X_2}$. Also find the marginal p.d.f. of Y_1, Y_2, and Y_3 and hence show that Y_1, Y_2, and Y_3 are independent.
Answer: The joint p.d.f. is

$$f_{(Y_1,Y_2,Y_3)}(y_1,y_2,y_3) = \begin{cases} y_1^2 y_2 e^{-y_1} & \text{if } 0 < y_1 < \infty, 0 < y_2, y_3 < 1 \\ 0 & \text{otherwise} \end{cases}$$

The marginal p.d.f. of Y_1 is

$$f_{Y_1}(y_1) = \begin{cases} \frac{1}{2} y_1^2 e^{-y_1} & \text{if } 0 < y_1 < \infty \\ 0 & \text{otherwise} \end{cases}$$

The marginal p.d.f. of Y_2 is

$$f_{Y_2}(y_2) = \begin{cases} 2y_2 & \text{if } 0 < y_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

The marginal p.d.f. of Y_3 is

$$f_{Y_3}(y_3) = \begin{cases} 1 & \text{if } 0 < y_3 < 1 \\ 0 & \text{otherwise} \end{cases}$$