Probability and Statistics
Problem Set I

1. Let E, F, G be three events. Find expressions for the events that of E, F, G

   (a) only F occurs

   (b) both E and F but not G occur

   (c) at least one event occurs

   (d) at least two event occur

   (e) all three events occur

   (f) none occurs

   (g) at most one event occurs

   (h) at most two events occur

   Answer: **(a)** $F \cap E^c \cap G^c$, **(b)** $E \cap F \cap G^c$, **(c)** $E \cup F \cup G$, **(d)** $(E \cap F) \cup (E \cap G) \cup (F \cap G)$, **(e)** $E \cap F \cap G$, **(f)** $(E \cup F \cup G)^c$, **(g)** $(E \cap F)^c \cap (E \cap G)^c \cap (F \cap G)^c$, **(h)** $(E \cap F \cap G)^c$

2. Let $\mathcal{S} = \{0, 1, 2, \cdots\}$ and $E \subseteq \mathcal{S}$. Then in each of the following cases, verify P is a probability on $\mathcal{S}$.

   (a) $P(E) = \sum_{x \in E} \frac{e^{-\lambda}\lambda^x}{!x}$, $\lambda > 0$.

   (b) $P(E) = \sum_{x \in E} p(1-p)^x$, $0 < p < 1$.

   (c) $P(E) = 0$, if E has finite number of elements, and $P(E) = 1$, if E has infinite number of elements.

   Answer: (a) Yes (b) Yes (c) No

3. Let $\mathcal{S}$ and $\Sigma = \mathfrak{B}_{\mathbb{R}}$. in each of the following cases, does P define a probability on $\mathcal{S}$?

   (a) For each interval I, let

   $$P(I) = \int_I \frac{1}{\pi(1+x^2)} dx$$

   (b) For each interval I, let $P(I) = 1$ if I is an interval of finite length, and $P(I) = 0$ if I is an infinite interval.

   (c) For each interval I, let $P(I) = 1$ if $I \subseteq (-\infty, 1)$ and $P(I) = \int_I (\frac{1}{2}) dx$ if $I \subseteq [1, \infty)$. (If $I = I_1 \cup I_2$, where $I_1 \subseteq (-\infty, 1)$ and $I_2 \subseteq [1, \infty)$, then $P(I) = P(I_2)$)

4. For events $E_1, E_2, \cdots, E_n$, show that

   (a) $P(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} P(E_i)$.
   This is known as Boole's inequality.

   (b) $P(\bigcap_{i=1}^{n} E_i) \geq \sum_{i=1}^{n} P(E_i) - (n-1)$.
   This is known as Bonferroni's inequality.

(c) $P(\bigcap_{i=1}^{n} E_i) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2)\cdots P(E_n|E_1 \cap E_2 \cap \cdots \cap E_{n-1})$, where $P(E_1 \cap E_2 \cap \cdots \cap E_{n-1}) > 0$.

5. Let E and F be two independent events. Then show that

   (a) E^c and F are independent.

   (b) E and F^c are independent.

   (c) E^c and F^c are independent.

6. Let E and F be two events such that $P(E) = p_1 > 0$, $P(F) = p_2 > 0$ and $p_1 + p_2 > 1$. Show that $P(F|E) \geq 1 - \frac{1-p_2}{p_1}$.

7. For any two events E and F, show that $P(E\cap F) - P(E)P(F) = P(E)P(F^c) - P(E\cap F^c) = P(E^c)P(F) - P(E^c \cap F) = P((E\cup F)^c) - P(E^c)P(F^c)$.

8. A cell-phone tower has a circular coverage area of radius 10 km. If a call is initiated from a random point in the coverage area, find the probability that the call comes from within 2 km of the tower.

9. What is the chance that a leap year, selected at random, will contain 53 Sundays?

   Answer: $\frac{2}{7}$.

10. A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy?

    Answer: $\frac{1}{3}$

11. Three numbers are chosen from 1 to 20. Find the probability that they are not consecutive.

    Answer: $\frac{187}{190}$

12. Consider two events A and B with $P(A) = 0.4$ and $P(B) = 0.7$. Determine the maximum and minimum possible values of $P(A\cap B)$ and the conditions under which each of these values is attained.

13. Two digits are chosen at random without replacement from the set of integers $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

    (a) Find the probability that both digits are greater than 5.

    (b) Show that the probability that the sum of the digits will be equal to 5 is the same as the probability that their sum will exceed 13.

    Answer: (a) $\frac{\binom{3}{2}}{\binom{8}{2}}$

14. Calculate the probability of drawing from a pack of cards one that is an ace or is a spade or shows an even number $(2, 4, 6, 8, 10)$.

    Answer: 31/52

15. A school contains students in grades $1, 2, 3, 4, 5,$ and 6. Grades $2, 3, 4, 5,$ and 6 all contain the same number of students, but there are twice of this number in grade 1. If a student is selected at random from a list of all the students in the school, what is the probability that she/he will be in grade 3?

16. Suppose that the blood test for some disease is reliable in the following sense: for people who are infected with the disease the test produces a positive result in 99.99% of cases; for people not infected a positive test result is obtained in only 0.02% of cases. Furthermore, assume that in the general population one person in $10,000$ people is infected. A person is selected at random and found to test positive for the disease. What is the probability that the individual is actually infected?

    Answer: $1/3$

17. Consider two independent fair coins tosses, in which all four possible outcomes are equally likely. Let $H_1 = \{\text{1st toss is a head}\}$, $H_2 = \{\text{2nd toss is a head}\}$, and $D = \{\text{the two tosses have different results}\}$. Find $P(H_1), P(H_2), P(H_1 \cap H_2), P(H_1|D), P(H_2|D)$, and $P(H_1 \cap H_2|D)$.

    Answer: $P(H_1) = \frac{1}{2} = P(H_2), P(H_1 \cap H_2) = \frac{1}{4}, P(H_1|D) = \frac{1}{2} = P(H_2|D)$, and $P(H_1 \cap H_2|D) = 0$.

18. There are two roads from A to B and two roads from B to C. Each of the four roads has probability p of being blocked by snow, independently of all the others. What is the probability that there is an open road from A to C?

19. Suppose that we have $n \geq 2$ letters and corresponding n addressed envelopes. If these letters are inserted at random in n envelopes, find the probability that no letter is inserted into the correct envelope.

    Answer: $\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n+2}\frac{1}{n!}$.

20. One ball is to be selected from a box containing red, white, blue, yellow, and green balls. If the probability that the selected ball will be red is $1/5$ and the probability that it will be white is $2/5$, what is the probability that it will be blue, yellow, or green?

21. Suppose that $n(\geq 3)$ persons $P_1, \cdots, P_n$ are made to stand in a row at random. Find the probability that there are exactly r persons between P_1 and P_2; here $r \in \{1, 2, \cdots, n-2\}$.

    Answer: $\frac{2(n-r-1)}{n(n-1)}$

22. Three numbers are chosen at random and without replacement from the set $\{1, 2, ..., 50\}$. Find the probability that the chosen numbers are in (a) arithmetic progression, and (b) geometric progression.

    Answer: (a) $\frac{600}{\binom{50}{3}}$ (b) (a) $\frac{44}{\binom{50}{3}}$

23. The organization that David Jones works for is running a father son dinner for those employees having at least one son. Each of these employees is invited to attend along with his youngest son. If Jones is known to have two children, what is the conditional probability that they are both boys given that he is invited to the dinner?

    Answer: $\frac{1}{3}$

24. An urn contains b black balls and r red balls. One of the balls is drawn at random, but when it is put back in the urn c additional balls of the same color are put in with it. Now suppose that we draw another ball. Find the probability that the first ball drawn was black given that the second ball drawn was red.

    Answer: $\frac{b}{b+r+c}$

25. In answering a question on a multiple-choice test a student either knows the answer or guesses. Let p be the probability that he knows the answer and $1-p$ the probability that he guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that she answered it correctly?

    Answer: $\frac{mp}{1+(m-1)p}$

26. There are three coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. When one of the three coins is selected at random and flipped, it shows heads. What is the probability that it was the two-headed coin?

    Answer: $\frac{4}{9}$

27. A conservative design team, call it C and an innovative design team, call it N, are asked to design a new product within a month. From the past experience we know that

    (a) the probability that team C is successful is $2/3$

    (b) the probability that team N is successful is $1/2$

    (c) the probability that at least one team is successful is $3/4$

    Assuming that exactly one successful design is produced, what is the probability that it was designed by team N?

    Answer: $\frac{1}{4}$

28. A box of fuses contains 20 fuses, of which 5 are defective. If 3 of the fuses are selected at random and removed from the box in succession without replacement, what is the probability that all three fuses are defective?

    Answer: $\frac{1}{114}$

29. There is a 1% probability for a hard drive to crash. Therefore, it has two backups, each having a 2% probability to crash, and all three components are independent of each other. The stored information is lost only in an unfortunate situation when all three devices crash. What is the probability that the information is saved?

    Answer: 0.99

30. A box contains 100 balls, of which r are red. Suppose that the balls are drawn from the box one at a time, at random, without replacement. Determine (a) the probability that the first ball drawn will be red; (b) the probability that the 50th ball drawn will be red; and (c) the probability that the last ball drawn will be red.

31. In screening for a certain disease, the probability that a healthy person wrongly gets a positive result is 0.05. The probability that a diseased person wrongly gets a negative result is 0.002. The overall rate of the disease in the population being screened is 1%. If the person X test gives a positive result, what is the probability that the person X actually have the disease?

    Answer: 0.168

32. Alice is taking a probability class and at the end of each week she can be either up-to-date or she may have fallen behind. If she is up-to-date in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.8 (or 0.2, respectively). If she is behind in a given week, the probability that she will be up-to-date (or behind) in the next week is

0.4 (or 0.6, respectively). Alice is (by default) up-to-date when she starts the class. What is the probability that she is up-to-date after three weeks?

Answer: 0.688

33. There are r players, with player i initially having n_i units, $n_i > 0, i = 1, \ldots, r$. At each stage, two of the players are chosen to play a game, with the winner of the game receiving 1 unit from the loser. Any player whose fortune drops to 0 is eliminated, and this continues until a single player has all $n = \sum_{i=1}^{r} n_i$ units, with that player designated as the victor. Assuming that the results of successive games are independent, and that each game is equally likely to be won by either of its two players, find the probability that player i is the victor.

    Answer: $\frac{n_i}{n}$

34. Consider an empty box in which four balls are to be placed (one-by-one) according to the following scheme. A fair die is cast each time and the number of spots on the upper face is noted. If the upper face shows up 2 or 5 spots then a white ball is placed in the box. Otherwise a black ball is placed in the box. Given that the first ball placed in the box was white find the probability that the box will contain exactly two black balls.

    Answer: $\frac{4}{9}$

35. On a midterm exam, students X, Y, and Z forgot to sign their papers. Professor knows that they can write a good exam with probabilities $0.8, 0.7$, and 0.5, respectively. After the grading, he notices that two unsigned exams are good and one is bad. Given this information, and assuming that students worked independently of each other, what is the probability that the bad exam belongs to student Z?

    Answer: 0.595