Problem Set- VI

1. Prove that
   (a) Every positive definite matrix is invertible.
   (b) The only positive definite projection matrix is identity matrix.

2. Prove that a positive definite matrix cannot have a zero (or a negative number) on its diagonal.
   (i) $\begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$, (ii) $\begin{bmatrix} b & 2 & 2 \\ 2 & b & 2 \\ 2 & 2 & b \end{bmatrix}$, (iii) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{bmatrix}$

3. Determine whether the following matrices are positive definite, positive semi definite, negative definite, negative semi definite, or indefinite.
   (i) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, (ii) $\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ -1 & -1 & -3 \end{bmatrix}$, (iii) $\begin{bmatrix} -2 & 4 & -1 \\ 4 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$, (iv) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & -2 \\ -1 & -2 & -4 \end{bmatrix}$.

4. Identify the following curves and surfaces
   (a) $16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0$
   (b) $3x^2 + 2y^2 + 3z^2 - 2zx + 2x - 2 = 0$

5. Find the singular values and singular value decomposition (SVD) of the following matrices:
   (i) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$, (ii) $\begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, (iii) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

6. Find the Jordan canonical form of the matrix satisfying the given conditions:
   (a) A is 3×3 with $p_A(\lambda) = (-2 - \lambda)^3$ and $m_A(\lambda) = (\lambda + 2)^2$.
   (b) A is 3×3 with $p_A(\lambda) = (2 - \lambda)^3$ and $m_A(\lambda) = (\lambda - 2)^2$.
   (c) A is 6×6 with $p_A(\lambda) = (5-\lambda)^3(4-\lambda)^2(3-\lambda)$ and $m_A(\lambda) = (\lambda-5)^2(\lambda-4)^2(\lambda-3)$. The matrix $A - 5I$ has rank 4.
   (d) A is 6×6 with $p_A(\lambda) = (5 - \lambda)^4(-4 - \lambda)^2$ and $m_A(\lambda) = (\lambda - 5)^2(\lambda + 4)^2$. The matrix $A - 5I$ has rank 3.

7. Find the Jordan canonical form J of the following matrices and find a non-singular matrix P with $P{-}1AP = J$: (i) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.