Problem Set-V

All notations are standard and are given explicitly in the last page of this sheet.

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1. Suppose V is a vector space over $\mathbb{F}$. Let $<,>: V \times V \longrightarrow \mathbb{F}$ be defined as follows:
   (a) $\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $\langle \tilde{x}, \tilde{y} \rangle = \tilde{x}\tilde{y}^t = x_1y_1 + x_2y_2 + \cdots + x_ny_n$.
   (b) $\langle , \rangle : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$ defined by $\langle \tilde{x}, \tilde{y} \rangle = \tilde{x}\tilde{y}^t$.
   (c) $\langle , \rangle : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$ defined by $\langle \tilde{x}, \tilde{y} \rangle = \tilde{x}\tilde{y}^*$.
   (d) $\langle , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $\langle \tilde{x}, \tilde{y} \rangle = x_1y_1 - 2x_1y_2 - 2y_1x_2 + 9x_2y_2$.
   (e) $\langle , \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ by $\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1y_1 + x_2y_1 + x_1y_2 + 2x_2y_2 + 3x_3y_2 + 3x_2y_3 + 9x_3y_3$.
   (f) $\langle , \rangle : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ defined by $\langle A, B \rangle = trace(AB^t)$.
   (g) $\langle , \rangle : P_1(\mathbb{R}) \times P_1(\mathbb{R}) \longrightarrow \mathbb{R}$ defined by $\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)dx$.

   Check whether the given function defines an inner product on V or not.

2. Let A be a 2×2 matrix with real entries. Define a map $\langle , \rangle$ from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}$ by $\langle (x_1, x_2), (y_1, y_2) \rangle = \begin{pmatrix} x_1 & x_2 \end{pmatrix} A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Show that $\langle , \rangle$ is an inner product on $\mathbb{R}^2$ iff $A = A^T$, $a_{11} > 0, a_{22} > 0$ and $\det(A) > 0$.

3. Let V be a real or complex vector space with an inner product. Show that $||x - y||^2 + ||x + y||^2 = 2||x||^2 + 2||y||^2$, for every $x, y \in V$. This is called parallelogram law.

4. (a) If V is a real inner product space, then for any $x, y \in V$, we have $\langle x, y \rangle = \frac{1}{4}(||x + y||^2 - ||x - y||^2)$.
   (b) If V is a complex inner product space, then for any $x, y \in V$, we have $\langle x, y \rangle = \frac{1}{4}(||x + y||^2 - ||x - y||^2 + i||x + iy||^2 - i||x - iy||^2)$.

5. Let V be a real inner product space.
   (a) Show that $x - y \perp x + y$ iff $||x|| = ||y||$ (The geometric meaning of this is that a parallelogram is a rhombus iff the diagonal are perpendicular).
   (b) Let V be a real inner product. Show that $x \perp y$ iff $||x - y||^2 = ||x||^2 + ||y||^2$ (This is Pythagoras theorem and its converse).
   (c) Show that if $||x + y|| = ||x|| + ||y||$, one is scalar multiple of the other.

6. Apply Gram-Schmidt process to obtain an orthonormal set:
   (a) $\{(-1, 0, 1), (1, -1, 0), (0, 0, 1)\}$ in $\mathbb{R}^3$ with usual inner product
   (b) $\{1, p_1(t) = t, p_2(t) = t^2\}$ of $\mathbb{P}_2(\mathbb{R})$ with inner product $\langle p, q \rangle = \displaystyle\int_0^1 p(t)q(t)dt$
   (c) $\{(1, -1, 1, -1), (5, 1, 1, 1), (2, 3, 4, -1)\}$ in $\mathbb{R}^4$ with usual inner product

7. Let $V = C([0,1])$ with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$. Find the orthogonal complement of the subspace of polynomial functions.

8. Let $V = M_n(\mathbb{C})$ with the inner product $\langle A, B\rangle = tr(AB^*)$. Find the orthogonal complement of the subspace of diagonal matrices.

9. Let W be a subspace of a finite dimensional inner product space V and $x \in V$ such that $\langle x, y\rangle + \langle y, x\rangle \leq \langle y, y\rangle$ for all $y \in W$. Show that $x \in W^\perp$.

10. Consider the subspace $W = \{(x, y, z, w) \mid x + 2y + z + w = 0 = x + y - 2z, w = 0\}$ of the standard inner product space $\mathbb{R}^4$. Find an orthonormal basis of W and $W^\perp$.

11. Consider $\mathbb{R}^4$ with the usual inner product. Let W be the subspace of $\mathbb{R}^4$ consisting of all vectors which are orthogonal to both $(1, 0, -1, 1)$ and $(2, 3, -1, 2)$. Find an orthonormal basis of W.

12. Find the projection of $v = (3+4i, 2-3i)$ along the vector $w = (5+i, 2i)$ in $\mathbb{C}^2$ over $\mathbb{C}$.

13. Suppose $W = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$. Find the shortest distance of $(a, b) \in \mathbb{R}^2$ from W with respect to i) the standard inner product, ii) the inner product defined by $\langle (x_1, y_1), (x_2, y_2)\rangle = 2x_1x_2 + y_1y_2$.

Note:

1. $\tilde{x}$- a vector in $\mathbb{F}^n$, i.e., $\tilde{x} = (x_1, x_2, \ldots, x_n)$.

2. A^T- transpose of a matrix A.

3. A^*- conjugate transpose of a matrix A.

4. $x \perp y$ means x is orthogonal to y i.e. $\langle x, y\rangle = 0$.

5. $W^\perp$ denotes orthogonal complement of W.

6. Let $f : [0,1] \longrightarrow \mathbb{R}$ be a continuous map such that $\int_0^1 x^n f(x)dx = 0$, for every $n \in \mathbb{N} \cup \{0\}$. Then $f(x) = 0$ for every $x \in [0,1]$.

7. The inner product defined by $\langle (x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$ is called usual inner product on $\mathbb{R}^n$.

8. The inner product defined by $\langle (z_1, z_2, \cdots, z_n), (w_1, w_2, \cdots, w_n)\rangle = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n$ is called usual inner product on $\mathbb{C}^n$.