Problem Set-4

1. Find the eigenvalues, eigenvectors and dimension of eigenspaces of the following operators.
(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (x + y, x)$.
(b) $T : \mathbb{C}^n \to \mathbb{C}^n$ defined by $T(x_1, x_2, \ldots, x_n) = (x_n, x_1, \ldots, x_{n-1})$.
(c) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(z_1, z_2) = (z_1 - 2z_2, z_1 + 2z_2)$.

2. Let A and B be two similar matrices. Then show that
   (a) A and B have same eigenvalues
   (b) Let λ be an eigenvalue of A (and hence of B). Then $AM_A(\lambda) = AM_B(\lambda)$ and $GM_A(\lambda) = GM_B(\lambda)$, where AM and GM denote algebraic multiplicity and geometric multiplicity respectively.

3. Let $A \in M_n(\mathbb{R})$ and λ be eigenvalue of A. Let $g(x) = a_0 + a_1 x + \cdots, +a_k x^k \in \mathbb{R}[x]$. Then $g(\lambda)$ is an eigenvalue of $g(A)$.

4. Let A be $m \times n$ matrix and B be $n \times m$ matrix with $n \geq m$. Then eigenvalues of BA are m eigenvalues of AB with more $(n - m)$ zero eigenvalues.

5. All non-zero eigenvalues of a skew hermitian matrix are purely imaginary.

6. Let A and B be two 4×4 real matrices with $-1, 2$ and 3 are three eigenvalues of $AB - BA$. Then find
   (a) determinant of $AB - BA$
   (b) determinant of $Adj(AB - BA)$.

7. Let A be a 2×2 real matrix and suppose that $A^2 = 0$. Show that for each $c \in \mathbb{R}$, $\det(cI_2 - A) = c^2$.

8. For any scalars a, b and c show that the matrices $A = \begin{bmatrix} b & c & a \\ c & a & b \\ a & b & c \end{bmatrix}$ and $B = \begin{bmatrix} c & a & b \\ a & b & c \\ b & c & a \end{bmatrix}$ are similar.

9. Let A be the 4×4 real matrix $\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$. Find the characteristic polynomial and the minimal polynomial of A.

10. Show that every matrix A such that $A^2 = A$ is similar to a diagonal matrix.

11. Find the eigenvalues and eigenvectors of the following matrices. Is A similar over $\mathbb{R}$ to a diagonal matrix? Is A similar over $\mathbb{C}$ to a diagonal matrix?
    (a) $A = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

12. Find a basis B such that $[T]_B$ is a diagonal matrix in case T is diagonalizable. Find P such that $[T]_B = P[T]_S P^{-1}$, where S is the standard basis in each case.
(a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (5x - 6y - 6z, -x + 4y + 2z, 3x - 6y - 4z)$.
(b) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(x, y) = (x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta)$.