Lecture 6

Vector Space and Its Properties

Definition 1. Let $\mathbb{F}$ be a field with binary operations $+$ (addition) and $\cdot$ (multiplication). A non empty set V is called a vector space over the field $\mathbb{F}$ if there exist two operations, called vector addition $\oplus$ and scalar multiplication $\odot$,

$$\oplus : V \times V \longrightarrow V \qquad \text{and} \qquad \odot : \mathbb{F} \times V \longrightarrow V,$$

such that the following conditions are satisfied.

1. Vector addition is associative, i.e., $v_1 \oplus (v_2 \oplus v_3) = (v_1 \oplus v_2) \oplus v_3$ for all $v_1, v_2, v_3 \in V$;
2. There is a unique vector $0 \in V$, called the zero vector, such that $v \oplus 0 = v = 0 \oplus v$ for all $v \in V$;
3. For each vector $v \in V$ there is a unique vector $-v \in V$ such that $v \oplus (-v) = 0$;
4. Vector addition is commutative, i.e., $v_1 \oplus v_2 = v_2 \oplus v_1$ for all $v_1, v_2 \in V$;
5. $\alpha \odot (v_1 \oplus v_2) = \alpha \odot v_1 \oplus \alpha \odot v_2$ for all $v_1, v_2 \in V$ and $\alpha \in \mathbb{F}$;
6. $(\alpha + \beta) \odot v = \alpha \odot v \oplus \beta \odot v$ for all $v \in V$ and $\alpha, \beta \in \mathbb{F}$;
7. $(\alpha \cdot \beta) \odot v = \alpha \odot (\beta \odot v)$ for all $v \in V$ and $\alpha, \beta \in \mathbb{F}$;
8. $1 \odot v = v$, where 1 is the multiplicative identity of the field $\mathbb{F}$.

If V is a vector space over the field $\mathbb{F}$, we denote it by $V(\mathbb{F})$. The elements of V are called **vectors** and elements of $\mathbb{F}$ are called **scalars**.

Example 2. *1.* $\mathbb{R}(\mathbb{R})$, $\mathbb{C}(\mathbb{C})$ *and* $\mathbb{C}(\mathbb{R})$ *are vector spaces under their usual addition and scalar multiplication.*

2. Let $V = \mathbb{F}^n = \{(x_1, \ldots, x_n) \,|\, x_1, \ldots, x_n \in \mathbb{F}\}$. *Then* V *forms a vector space over* $\mathbb{F}$ *under the following operations:*

$$(x_1, \ldots, x_n) \oplus (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n),$$

$$\alpha \odot (x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in V$ *and* $\alpha \in \mathbb{F}$.

3. The set of all $m \times n$ *matrices* $M_{m\times n}(\mathbb{F})$ *with entries from the field* $\mathbb{F}$ *is a vector space over the field* $\mathbb{F}$ *under the following operations:*

$$(a_{ij}) \oplus (b_{ij}) = (a_{ij} + b_{ij}), \text{ and } \alpha \odot (a_{ij}) = (\alpha a_{ij}),$$

for all $A = (a_{ij}), B = (b_{ij}) \in M_{m\times n}(\mathbb{F})$ *and* $\alpha \in \mathbb{F}$.

4. Let X *be a non-empty set. Let* V *be the set of all the functions from* X *to* $\mathbb{R}$. *Then* V *forms a vector space over* $\mathbb{R}$ *under the following operations:* $(f \oplus g)(x) = f(x) + g(x)$ *and* $(\alpha \odot f)(x) = \alpha f(x)$, *for all* $x \in X$, $f, g \in V$, *and* $\alpha \in \mathbb{R}$.

5. Let $P_n = \{a_0 + a_1x + \ldots + a_nx^n \mid a_0, a_1, \ldots, a_n \in \mathbb{F}\}$. *The set* P_n *forms a vector space over* $\mathbb{F}$ *under the following operations:*

$$(a_0 + a_1x + \cdots + a_nx^n) \oplus (b_0 + b_1x + \ldots + b_nx^n) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

$$\alpha \odot (a_0 + a_1x + \cdots + a_nx^n) = (\alpha a_0 + \alpha a_1x + \cdots + \alpha a_nx^n)$$

for all $(a_0 + a_1x + \cdots + a_nx^n), (b_0 + b_1x + \cdots + b_nx^n) \in P_n$ *and* $\alpha \in \mathbb{F}$.

6. $\mathbb{R}^2$ *over* $\mathbb{R}$ *is not a vector space with respect to the following operations*

$$\begin{aligned}(x_1, y_1) \oplus (x_2, y_2) &= (x_1 + x_2 + 1, y_1 + y_2 + 1)\\ \alpha \odot (x, y) &= (\alpha x, \alpha y),\end{aligned}$$

where $(x_1, y_1), (x_2, y_2), (x, y) \in \mathbb{R}^2$ *and* $\alpha \in \mathbb{R}$. *To see this, we need to find which property is not satisfied. Let* $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ *and* $\alpha \in \mathbb{R}$. *Then*

$$\begin{aligned}\alpha \odot ((x_1, y_1) \oplus (x_2, y_2)) &= \alpha \odot (x_1 + x_2 + 1, y_1 + y_2 + 1)\\ &= (\alpha(x_1 + x_2 + 1), \alpha(y_1 + y_2 + 1))\\ &= (\alpha x_1 + \alpha x_2 + \alpha, \alpha y_1 + \alpha y_2 + \alpha)\\ &\neq \alpha \odot (x_1, y_1) \oplus \alpha \odot (x_2, y_2)\end{aligned}$$

Take $\alpha = 2$ *and* $(x_1, y_1) = (1, 1) = (x_2, y_2)$.

Remark 3. If $\mathbb{F}_1$ is a subfield of $\mathbb{F}$, then $\mathbb{F}(\mathbb{F}_1)$ forms a vector space but converse is not true. For example, $\mathbb{C}(\mathbb{R})$ is a vector space but $\mathbb{R}(\mathbb{C})$ is not a vector space.

Note: If there is no confusion between the operations on a vector space and the operations on the field, we simply write $\oplus$ by $+$ and $\odot$ by $\cdot$.

Theorem 4. Let V be a vector space over $\mathbb{F}$. Then

1. $0 \cdot \boldsymbol{v} = \mathbf{0}$, where 0 and $\mathbf{0}$ are additive identity of $\mathbb{F}$ and V respectively, and $\boldsymbol{v} \in V$.

2. $\alpha \cdot \mathbf{0} = \mathbf{0} \quad \forall \alpha \in \mathbb{F}$.

3. $(-1) \cdot \boldsymbol{v} = -\boldsymbol{v}$.

4. if $\alpha \in \mathbb{F}$ and $\boldsymbol{v} \in V$ such that $\alpha \cdot \boldsymbol{v} = \mathbf{0}$, then either $\alpha = 0$ or $\boldsymbol{v} = \mathbf{0}$.

Proof: For the first statement, we write $0 = 0 + 0$ so that

$$\begin{aligned}
0 \cdot \boldsymbol{v} &= (0+0) \cdot \boldsymbol{v} \\
0 \cdot \boldsymbol{v} &= 0 \cdot \boldsymbol{v} + 0 \cdot \boldsymbol{v} \quad \text{(Condition 6.)} \\
0 \cdot \boldsymbol{v} + (-0 \cdot \boldsymbol{v}) &= 0 \cdot \boldsymbol{v} + 0 \cdot \boldsymbol{v} + (-0 \cdot \boldsymbol{v}) \quad \text{(using additive inverse)} \\
0 \cdot \boldsymbol{v} + (-0 \cdot \boldsymbol{v}) &= 0 \cdot \boldsymbol{v} + (0 \cdot \boldsymbol{v} + (-0 \cdot \boldsymbol{v})) \quad \text{(using additive inverse and additive associativity)} \\
\mathbf{0} &= 0 \cdot \boldsymbol{v} + \mathbf{0} = 0 \cdot \boldsymbol{v}.
\end{aligned}$$

For the second statement, write $\mathbf{0} = \mathbf{0} + \mathbf{0}$ so that

$$\begin{aligned}
\alpha \cdot \mathbf{0} &= \alpha \cdot (\mathbf{0} + \mathbf{0}) \\
\alpha \cdot \mathbf{0} &= \alpha \cdot \mathbf{0} + \alpha \cdot \mathbf{0} \quad \text{(Condition 5.)} \\
\alpha \cdot \mathbf{0} + (-\alpha \cdot \mathbf{0}) &= \alpha \cdot \mathbf{0} + \alpha \cdot \mathbf{0} + (-\alpha \cdot \mathbf{0}) \quad \text{(using additive inverse)} \\
\alpha \cdot \mathbf{0} + (-\alpha \cdot \mathbf{0}) &= \alpha \cdot \mathbf{0} + (\alpha \cdot \mathbf{0} + (-\alpha \cdot \mathbf{0})) \quad \text{(using additive inverse and additive associativity)} \\
\mathbf{0} &= \alpha \cdot \mathbf{0} + \mathbf{0} = \alpha \cdot \mathbf{0}.
\end{aligned}$$

For the third statement, we write $0 = (-1) + 1$ so that

$$\begin{aligned}
0 \cdot \boldsymbol{v} &= ((-1) + (1)) \cdot \boldsymbol{v} \\
0 \cdot \boldsymbol{v} &= (-1) \cdot \boldsymbol{v} + 1 \cdot \boldsymbol{v} \quad \text{(using Condition 5.)} \\
\mathbf{0} &= (-1) \cdot \boldsymbol{v} + \boldsymbol{v} \quad \text{(using the first statement and Condition 8.)} \\
\mathbf{0} + (-\boldsymbol{v}) &= (-1) \cdot \boldsymbol{v} + (\boldsymbol{v} + (-\boldsymbol{v})) \quad \text{(ussing additive inverse and associativity)} \\
-\boldsymbol{v} &= (-1) \cdot \boldsymbol{v} + \mathbf{0} = (-1) \cdot \boldsymbol{v}.
\end{aligned}$$

Prove the fourth statement yourself.

Definition 5. Let V be a vector space over the field $\mathbb{F}$. A subspace of V is a non-empty subset W of V which is itself a vector space over $\mathbb{F}$ with the operations of vector addition and scalar multiplication on V.

Example The subsets $\{\mathbf{0}\}$ and V are subspaces of a vector space V. These subspaces are called trivial subspaces of V.

Theorem 6. Let V be a vector space over the field $\mathbb{F}$ and $W \subseteq V$. Then W is subspace of V if and only if $\alpha w_1 + \beta w_2 \in W$, for all $w_1, w_2 \in W$ and $\alpha, \beta \in \mathbb{F}$.

Proof: Direct part follows from the definition of subspace. Conversely, if $\alpha = 1$ and $\beta = 1$, then we see that $w_1 + w_2 \in W \; \forall w_1, w_2 \in W$, also if $\beta = 0$, then $\alpha w_1 \in V \; \forall \alpha \in \mathbb{F}$ and $w_1 \in W$. Thus, W is closed under vector addition and scalar multiplication. Further, let $\alpha = \beta = -1$ and $w_1 = w_2$. Then $0 \in W$, *i.e.*, zero vector of V lies in W. The rest of the properties trivially true as the elements are from vector space V. Thus, W is a vector space over $\mathbb{F}$. $\square$

Example 7. 1. A line passing through origin is a subspace of $\mathbb{R}^2$ over $\mathbb{R}$.

2. Let A be an $m \times n$ matrix over $\mathbb{F}$. Then the set of all $n \times 1$ (column) matrices x over $\mathbb{F}$ such that $Ax = 0$ is a subspace of the space of all $n \times 1$ matrices over $\mathbb{F}$ or $\mathbb{F}^n$. To see this we need to show that $A(\alpha x + y) = 0$, when $Ax = 0, Ay = 0$, and α is an arbitrary scalar in $\mathbb{F}$.

3. The solution set of a system of non-homogeneous linear equations is not a subspace of $\mathbb{F}^n$ over $\mathbb{F}$.

4. The collection of polynomial of degree less than or equal to n over $\mathbb{R}$ with the constant term 0 forms a subspace of the space of polynomials of degree less than or equal to n.

5. The collection of polynomial of degree n over $\mathbb{R}$ is not a subspace of the space of polynomials of degree less than or equal to n.

Theorem 8. Let W_1 and W_2 be subspaces of a vector space V over $\mathbb{F}$. Then $W_1 \cap W_2$ is a subspace of V.

Proof: Since W_1 and W_2 are subspaces, $0 \in W_1 \cap W_2$ so that $W_1 \cap W_2$ is a non-empty set. Let $w, w' \in W_1 \cap W_2$ and $\alpha, \beta \in \mathbb{F}$. Then $\alpha w + \beta w' \in W_1$ as W_1 is a subspace of V and $w, w' \in W_1$. Similarly, $\alpha w + \beta w' \in W_2$. Thus, $\alpha w + \beta w' \in W_1 \cap W_2$. By Theorem 6, $W_1 \cap W_2$ is a subspace of V.

Remark 9. The above theorem can be generalized for any number of subspaces. However, the union of two subspaces need not be a subspace. Let $V = \mathbb{R}^2$, $W = X$-axis and $W' = Y$-axis. Then $(1,0) \in W$ and $(0,1) \in W'$ but $(1,0) + (0,1) = (1,1) \notin W \cup W'$. The union of two subspaces is a subspace if one is contained in other.