Lecture 4
Invertible Matrix & Gauss-Jordan Method

Definition 1. Invertible Matrix: *A square matrix M is said to be* **invertible** *if there exists a matrix N of the same order such that $MN = NM = I$. The matrix N is called inverse of M and is denoted as M^{-1}.*

Theorem 2. Let A and B be two $n \times n$ matrices then: (a) if A is invertible, then so is A^{-1} with $(A^{-1})^{-1} = A$; (b) if both A and B are invertible, then so is AB with $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem 3. An elementary matrix is invertible.

Proof: Let E be an elementary matrix corresponding to the elementary row operation ρ. If ρ' is the inverse operation of ρ and $E' = \rho'(I)$, then $EE' = \rho(I)\rho'(I) = \rho(\rho'(I)) = (\rho \circ \rho')(I) = I$ and $E'E = \rho'(I)\rho(I) = \rho'(\rho(I)) = (\rho' \circ \rho)(I) = I$ so that E is invertible. □

Theorem 4. Let A be an $m \times n$ matrix. Then by applying a sequence of row and column operations A can be reduced to the form

$$\begin{bmatrix} I_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}_{m\times n}$$

which is called the **normal form** of the matrix, equivalently, there exist elementary row matrices $E_1, \ldots, E_s$, and elementary column matrics $F_1, \ldots, F_k$ such that

$$E_1 \cdots E_s A F_1 \cdots F_k = \begin{bmatrix} I_{r\times r} & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(n-r)} \end{bmatrix}.$$

Theorem 5. Let A be an $n \times n$ matrix. Then A is invertible if and only if A is a product of elementary matrices.

Proof: If A is an invertible matrix then there exist elementary matrices $E_1, \ldots, E_s, F_1, \ldots, F_k$ such that

$$E_1 \cdots E_s A F_1 \cdots F_k = \begin{bmatrix} I_{r\times r} & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(n-r)} \end{bmatrix} = I_n.$$

Therefore, $A = E_s^{-1} \ldots E_1^{-1} I_n \ F_k^{-1} \ldots F_1^{-1}$. Note that an elementary column matrix is one of the elementary row matrices. Further, inverse of an elementary matrix is again an elementary matrix. Hence, A is a product of elementary matrices. Converse follows from the fact that the product of invertible matrices is invertible. □

Theorem 6. Let A be an $n \times n$ matrix. Then A is invertible if and only if A can be reduced to the identity matrix I_n by performing a finite sequence of elementary row operations on A.

Proof: If A is invertible then by above theorem $A = E_k \cdots E_1$ for some $k \in \mathbb{N}$, equivalently $E_1^{-1} \cdots E_k^{-1} A = I$. Thus A can be reduced to identity matrix. Conversely, if A can be reduced to the identity matrix I_n by performing a finite sequence of elementary row operations on A. Then there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_1 A = I$, then $A = E_1^{-1} \cdots E_s^{-1}$. Therefore, A is invertible as product of invertible matrices is invertible.

Gauss-Jordan Method for finding inverse: Let A be an invertible matrix. Then there exist elementary matrices E_1, E_2, …, E_k such that $I = E_k E_{k-1} \ldots E_1 A$ which is equivalent to $A^{-1} = E_k E_{k-1} \ldots E_1 I$. This shows that sequence of elementary operations which reduces A to the identity matrix I, also reduces I to A^{-1} by performing in the same order.

Example 7. *Find inverse of* $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$ *by using Gauss-Jordan method.*

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \overset{R_2 \to R_2 - R_1, R_3 \to R_3 - R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\overset{R_3 \to R_3 - R_2, R_1 \to R_1 - R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right) \overset{R_3 \to R_3/2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right)$$

$$\overset{R_1 \to R_1 - R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1/2 & -1/2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right) = (I \mid A^{-1})$$

Therefore, $A^{-1} = \begin{pmatrix} 2 & -1/2 & -1/2 \\ -1 & 1 & 0 \\ 0 & -1/2 & 1/2 \end{pmatrix}$.

Gauss-Jordan elimination method for finding solutions of a system of linear equations Let $AX = B$ be a system of linear equations. Now consider the augmented matrix $(A|B)$. Apply finite number of elementary row operations to get the form $(A'|B')$. Here $(A'|B')$ is row reduced echelon form of the matrix $(A|B)$. Thus $(A'|B')$ is row equivalent to $(A|B)$, therefore $AX = B$ and $A'X = B'$ are equivalent systems and hence they have the same solution.

Example 2: Solve the following system of linear equations

$$x + 3y + z = 9$$
$$x + y - z = 1$$
$$3x + 11y + 5z = 35.$$

Solution: $(A|B) = \left(\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{array}\right) \overset{R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1}{\sim} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 2 & 8 \end{array}\right)$

$$\overset{R_3 \to R_3 - R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{R_2 \to -R_2/2}{\sim} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{R_1 \to R_1 - 3R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) = (A'|B').$$

The equivalent system is

$$x - 2z = -3$$
$$y + z = 4.$$

The solution set is $\{(2z - 3, 4 - z, z) : z \in \mathbb{R}\}$.

Definition 8. *A system of linear equation $Ax = b$ is said to be consistent if it has at least one solution (unique or infinitely many) and the system is called inconsistent if it has no solution.*

Theorem 9. *Consider a system of linear equation $Ax = b$, where $A \in M_{m\times n}(\mathbb{R})$. Suppose R and $(R|b')$ are the RRE forms of A and $(A|b)$ respectively. Let r and r' be the number of non-zero rows in R and $(R|b)$. Then*

1. *if $r \neq r'$, the system is inconsistent.*
2. *if $r = r' = n$, the system the unique solution.*
3. *if $r = r' < n$, the system has infinitely many solutions.*

Proof. Case 1: Note that $r' \geq r$. If $r \neq r'$, then $(R|b')_{r+1,n+1} = 1$ whereas $(R|b')_{r+1,j} = 0$ for all $j < n+1$. Suppose the system $Ax = b$ is consistent and y is one of its solutions. Then y is a solution of $Rx = b'$ (row-equivalent systems). The $r+1$-th equation of $Rx = b'$ gives that $0 = 1$, which is absurd, hence the system has no solution, that is, the system is inconsistent.

Case 2: If $r = r' = n$, then $(R|b') = \left(\begin{array}{c|c} I_n & b''_{n\times 1} \\ 0_{m-n\times n} & 0_{m-n\times 1} \end{array}\right)$. Therefore, $x = b''$ is the only solution of the system $Ax = b$.

Case 3: If $r = r' < n$, then $(R|b') = \left(\begin{array}{c|c} R'_{r\times n} & b''_{r\times 1} \\ 0_{m-r\times n} & 0_{m-r\times 1} \end{array}\right)$ so that the system $Rx = b'$ is equivalent to the system $R'x = b''$ for which the number of equations is less than the number of variables. Thus, $R'x = b''$ has infinitely many solutions and so $Rx = b'$ as well as $Ax = b$. □

Example 3: Find $a, b \in \mathbb{R}$ such that the following system of equations (i) is consistent, and (ii) is inconsistent (iii) has a unique solution (iv) has infinitely many solutions.

$$x + ay = 1, 2x + y = b.$$

The augmented matrix of the system is $\left(\begin{array}{cc|c} 1 & a & 1 \\ 2 & 1 & b \end{array}\right)$. Thus,

$$\left(\begin{array}{cc|c} 1 & a & 1 \\ 2 & 1 & b \end{array}\right) \overset{R_2 \to R_2 - 2R_1}{\sim} \left(\begin{array}{cc|c} 1 & a & 1 \\ 0 & 1-2a & b-2 \end{array}\right)$$

Case 1: If $1 - 2a = 0$ and $b - 2 \neq 0$. Then, the RRE form is $\left(\begin{array}{cc|c} 1 & a & 1 - \frac{1}{b-2} \\ 0 & 0 & 1 \end{array}\right)$. Thus, $r = 1$ and $r' = 2$. Therefore, the system has no solution (system is inconsistent).

Case 2: If $1 - 2a = 0$ and $b - 2 = 0$. Then, the RRE form is $\left(\begin{array}{cc|c} 1 & a & 1 - \frac{1}{b-2} \\ 0 & 0 & 0 \end{array}\right)$. Thus, $r = r' = 1 < 2$. Therefore, the system has infinitely many solutions.

Case 3: If $1 - 2a \neq 0$ and $b \in \mathbb{R}$. Then, the RRE form is $\left(\begin{array}{cc|c} 1 & 0 & 1 - a\frac{b-2}{1-2a} \\ 0 & 1 & \frac{b-2}{1-2a} \end{array}\right)$. Thus, $r = r' = 2$. Therefore, the system has unique solution.

Hence,

(i) the system is consistent when either $a \neq 1/2$, and $b \in \mathbb{R}$ or $a = 1/2$ and $b = 2$.

(ii) the system is inconsistent when $a = 1/2$ and $b \neq 2$.

(iii) the system has a unique solution if $a \neq 1/2$ and $b \in \mathbb{R}$.

(iv) the system has infinitely many solutions if $a = 1/2$ and $b = 2$.