Lecture 24
Jordan-Canonical Form

We know that not every matrix is similar to a diagonal matrix. Here, we discuss the simplest matrix to which a square matrix is similar. This simplest matrix coincides with a diagonal matrix if the matrix is diagonalizable.

Definition 1. *A square matrix A is called* **block diagonal** *if A has the form*

$$\begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & A_k \end{pmatrix},$$

where A_i is a square matrix and the diagonal entries of A_i lie on the diagonal of A.

Definition 2. *Let $\lambda \in \mathbb{C}$. A* **Jordan block** *$J(\lambda)$ is an upper triangular matrix whose all diagonal entries are λ, all entries of the superdiagonal (entries just above the diagonal) are 1 and other entries are zero. Therefore,*

$$J(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix}.$$

Definition 3. *A* **Jordan form** *or* **Jordan-Canonical form** *is a block diagonal matrix whose each block is a Jordan block, that is, Jordan form is a matrix of the following form*

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & J_k \end{pmatrix}.$$

Definition 4. *Let $T : V \to V$ be a linear transformation and $\lambda \in \mathbb{C}$. A non-zero vector $v \in V$ is called a generalized eigenvector of T corresponding to Λ if $(T - \lambda I)^p(v) = 0$ for some positive integer p.*

The generalized eigenspace of T corresponding to λ, denoted by K_λ, is the subset of V defined by

$$K_\lambda = \{v \in V \mid (T - \lambda I)^p(v) = 0 \text{ for some natural number } p\}.$$

Remark 5. *1. If $v \in V$ is a generalized eigenvector of a linear transformation T corresponding to $\lambda \in \mathbb{C}$, then λ is an eigenvalue of T.*
2. The generalized eigenspace K_λ is a subspace of V and $Tx \in K_\lambda$ for all $x \in K_\lambda$.
3. Let E_λ be the eigenspace corresponding to λ. Them $E_\lambda \subset K_\lambda$.

Theorem 6. *Let J be an $m \times m$ Jordan block with eigenvalue λ. Then characteristic polynomial of J is equal to its minimal polynomial, that is $p_J(x) = (x - \lambda)^m = m_J(x)$.*

Proof. Note that J is an upper-triangular matrix, hence the characteristic polynomial is $(x-\lambda)^m$ and the minimal polynomial is $(x-\lambda)^k$ for some $1 \le k \le m$. Here, we claim that $(J-\lambda I)^k \neq 0$ for $k < m$. Observe that, $J - \lambda I = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$, $(J-\lambda I)^2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$ and $(J-\lambda I)^{m-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$ so that the minimal polynomial of J is $(x-\lambda)^m$. □

Remark 7. *1. If a matrix A is similar to a Jordan block of order m with eigenvalue λ, then there exist an invertible matrix P such that $P^{-1}AP = J$. Let X_i be the i-th column of P. Then $\{X_1, X_2, \dots, X_m\}$ is a basis of $\mathbb{R}^m$, which is called* **Jordan basis***.*
2. The vector X_1 is an eigenvector corresponding to λ and $X_{j-1} = (A-\lambda)X_j$ for $j = 2, \dots, m$.

Theorem 8. *An $m \times m$ matrix A is similar to an $m \times m$ Jordan block J with eigenvalue λ if and only if there exist m independent vectors $X_1, X_2, \dots, X_m$ such that $(A-\lambda I)X_1 = 0$, $(A-\lambda I)X_2 = X_1, \dots, (A-\lambda I)X_m = X_{m-1}$.*

Example 9. *Consider $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$. Then the characteristic polynomial and minimal polynomial are the same which is $(x-2)^2$. Hence the matrix in not diagonalizable. Here, $(1,-1)$ is an eigenvector corresponding to 2. If J is the Jordan form of A, then we have a basis $\{X_1, X_2\}$ with respect to which the matrix representation of A is J. By previous theorem X_1 is an eigenvector and X_2 can be found by solving $(A-2I)X_2 = X_1$. Set $X_1 = (1,-1)$, then $X_2 = (1,0)$. Now construct $P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and verify that $P^{-1}AP = J$, where $J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.*

Theorem 10. *Let A be an $n \times n$ matrix with the characteristic polynomial $(x-\lambda_1)^{r_1} \cdots (x-\lambda_k)^{r_k}$, where λ_i's are distinct. Then A is similar to a matrix of the following form*

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & J_k \end{pmatrix},$$

where $J_1, J_2, \dots, J_k$ are Jordan blocks. The matrix J is unique except for the order of the blocks $J_1, J_2, \dots, J_k$.

Remark 11. *1. The sum of orders of the blocks corresponding to λ_i is r_i (the A.M.(λ_i)).*
2. The order of the largest block associated to λ_i is s_i, the exponent of $x-\lambda_i$ in the minimal polynomial of A.
3. The number of blocks associated with the eigenvalue λ_i is equal to the $GM(\lambda_i)$.
4. Knowing the characteristic polynomial and the minimal polynomial and the geometric multiplicity of each eigenvalue λ_i need not be sufficient to determine Jordan form of a matrix.

Example 12. *Let A be a matrix with characteristic polynomial $(x-1)^3(x-2)^2$ and minimal polynomial $(x-1)^2(x-2)$. Then we can find the Jordan form J of A by using above remarks,*
(i) The eigenvalue 1 appears on the diagonal 3 times, and 2 appears 2 times.
(ii) The largest Jordan block corresponding to $\lambda = 1$ is of order 2 (exponent of $(x-1)$ in the minimal polynomial), and the largest Jordan block corresponding to $\lambda = 2$ is of order 1.
(iii) The number number of Jordan blocks corresponding to $\lambda = 1$ is 2 where one block is of order 2 and other is of order 1. (iv) The number number of Jordan blocks corresponding to $\lambda = 2$ is 2 where both the blocks are of order 1. Therefore, the Jordan form of A is

$$\begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & & & \\ & (1) & & \\ & & (2) & \\ & & & (2) \end{pmatrix}.$$

Example 13. *Let A be a matrix with characteristic polynomial $(x-1)^3(x-2)^2$ and minimal polynomial $(x-1)^3(x-2)^2$. Then*
(i) The eigenvalue 1 appears on the diagonal 3 times, and 2 appears 2 times.
(ii) The largest Jordan block corresponding to $\lambda = 1$ is of order 3 (exponent of $(x-1)$ in the minimal polynomial), and the largest Jordan block corresponding to $\lambda = 2$ is of order 2.
(iii) The number number of Jordan blocks corresponding to $\lambda = 1$ is 1. (iv) The number number of Jordan blocks corresponding to $\lambda = 2$ is 1. Therefore, the Jordan form of A is

$$\begin{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \\ & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}.$$

Example 14. minimal and characteristic are not always sufficient *Let A be a matrix with characteristic polynomial $(x-1)^4$ and minimal polynomial $(x-1)^2$. Then*
(i) The eigenvalue 1 appears on the diagonal 4 times.
(ii) The largest Jordan block corresponding to $\lambda = 1$ is of order 2 (exponent of $(x-1)$ in the minimal polynomial).
(iii) The number number of Jordan blocks corresponding to $\lambda = 1$ is GM(1) which is not known. Note that $GM(1) \leq 4$ as minimal polynomial confirms that A is not diagonalizable. Also, $GM(1) \neq 1$, if $GM(1) = 1$, the the Jordan matrix has only one block corresponding to $\lambda = 1$ which must be of order 4, which is not true.
(iv) Thus $GM(1) = 2$ or 3.

(v) If $GM(1) = 2$ the Jordan form of A is

$$\begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \\ & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

(vi) If $GM(1) = 3$, the Jordan form of A is

$$\begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & & \\ & (1) & \\ & & (1) \end{pmatrix}.$$

Example 15. Possible Jordan forms for a given characteristic polynomial *Let A be a matrix with characteristic polynomial $(x-1)^3(x-2)^2$. Then choices of minimal polynomials are*
(i)$(x-1)(x-2)$, then Jordan form is the diagonal matrix.
(ii) $(x-1)^2(x-2)$, Example 12.
(iii) $(x-1)^3(x-2)$, the Jordan form is $\begin{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & & \\ & (2) & \\ & & (2) \end{pmatrix}$.
(iv) $(x-1)(x-2)^2$, $\begin{pmatrix} (1) & & & \\ & (1) & & \\ & & (1) & \\ & & & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}$.
(v) $(x-1)^2(x-2)^2$, $\begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & & \\ & (1) & \\ & & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}$.
(vi) $(x-1)^3(x-2)^2$, Example 13.

Example 16. (minimal, characteristic and $GM(\lambda)$ are not always sufficient) *Let A be a matrix with characteristic polynomial $(x-1)^7$ and minimal polynomial $(x-1)^3$ and $GM(1) = 3$. Then there are two possible Jordan forms (write the corresponding Jordan forms yourself!):*
(i) One Jordan block of order 3 and other two blocks of order 2.
(ii) Two Jordan blocks of order 3 and one of order 1.

Example 17. Find a Jordan basis *Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. The characteristic polynomial of A is $(x-1)^3$ and $A - I = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. Thus nullity$(A-I)$ is 1=GM(1). Therefore, the Jordan form of A is $J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. The problem is to find a Jordan basis or a matrix P such that $P^{-1}AP = J$. $P = [X_1\, X_2\, X_3]$, where $(A-I)X_1 = X_1$, $(A-I)X_2 = X_1$, $(A-I)X_3 = X_2$. On solving, we get $X_1 = (1, 0, -1)$, $X_2 = (1, 1, -1)$ or $(-1, 1, 1)$ and $X_3 = (1, 1, 0)$ or $(0, 1, 1)$. Hence, $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix}$.*

Example 18. *(***Finding a Jordan basis is not always straight forward***)*

Let $A = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$*. The characteristic polynomial of* A *is* $(x-2)^3$ *and* $A - 2I = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$*.*

Thus nullity$(A - I2)$ *is 2=GM(2). Therefore, the Jordan form of* A *is* $J = \begin{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & \\ & (2) \end{pmatrix}$*. The problem is to find a Jordan basis or a matrix* P *such that* $P^{-1}AP = J$*.* $P = [X_1\, X_2\, X_3]$*. Here, we get an eigenvector* (x, y, z) *satisfies* $x + y = 0$*, two independent eigenvectors are* $(0, 0, 1)$ *and* $(-1, 1, 0)$*. Note that each eigenvector corresponds to a Jordan block. Thus, set* $X_1 = (0, 0, 1)$*,* $(A - 2I)X_2 = X_1$*,* $X_3 = (-1, 1, 0)$ *or . But,* $(A - 2I)X_2 = X_1 \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ *which is an inconsistent system. Similarly,* $(A - 2I)X_2 = X_1$*, where* $X_1 = (-1, 1, 0)$ *is inconsistent. For finding a Jordan basis, we will change the eigenvector, let* $X_1 = (-1, 1, -1)$*, then* $X_2 = (0, -1, 0)$*, and* $X_3 = (-1, 1, 0)$ *Hence,* $P = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$*.*