## Lecture 22
## Positive & Negative Definite Matrices & Singular Value Decomposition(SVD)

**Definition 1.** *Let $A$ be a real symmetric matrix. Then $A$ is said to be positive (negative) definite if all of its eigenvalues are positive (negative).*

**Definition 2.** *Let $A$ be a real symmetric matrix. Then $A$ is said to be positive (negative) semi-definite if all of its eigenvalues are non-negative (non-positive).*

**Remark 3.** *1. If $A$ is positive definite, then $\det(A) > 0$ and $tr(A) > 0$.*
*2. If $A$ is negative definite matrix of order $n$, then $tr(A) < 0$. If $n$ is even, $\det(A) > 0$ and if $n$ is odd $\det(A) < 0$.*
*3. If $A$ is positive semi-definite, then $\det(A) \geq 0$ and $tr(A) \geq 0$.*
*4. If $A$ is negative semi-definite matrix of order $n$, then $tr(A) \leq 0$. If $n$ is even, $\det(A) \geq 0$ and if $n$ is odd $\det(A) \leq 0$.*

**Proposition 4.** *Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Then*
*1. $A$ is positive definite if and only if $X^TAX > 0$ for all $0 \neq X \in \mathbb{R}^n$.*
*2. $A$ is negative definite if and only if $X^TAX < 0$ for all $0 \neq X \in \mathbb{R}^n$.*

*Proof.* Let $A$ be positive definite. Since $A$ is a real symmetric matrix, $A$ is orthogonally diagonalizable with positive eigenvalues. Therefore, $A = PDP^T$, where $D$ is a diagonal matrix with entries as eigenvalues of $A$ and $P$ is an orthogonal matrix. Thus, $X^TAX = X^TPDP^TX = (P^TX)^TD(P^TX) = Y^TDY$, where $Y = P^TX \neq 0$. Let $Y = (y_1, y_2, \ldots, y_n)^T$. Then $X^TAX = Y^TDY = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 > 0$, where $\lambda_i$ are eigenvalues of $A$.

Conversely, let $X^TAX > 0$ for all $X \in \mathbb{R}^n$. Let $\lambda \in \mathbb{R}$ be an eigenvalue of $A$ and $X_0$ be an eigenvector corresponding to $\lambda$. Then $X_0^TAX_0 > 0 \Rightarrow \lambda X_0^TX_0 > 0$. Note that $X_0^TX_0 = \|X_0\|^2 > 0$ as $X_0 \neq 0$. Therefore, $\lambda > 0$. □

**Proposition 5.** *Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Then*
*1. $A$ is positive definite if and only if $A = B^TB$ for some invertible matrix $B$.*
*2. $A$ is positive semi-definite if and only if $A = B^TB$ for some matrix $B$.*

*Proof.* Let $A$ be a positive definite matrix. Then $A$ is symmetric, by Spectral theorem, there exists an orthogonal matrix $P$ such that $P^TAP = D$ with $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_i$'s are eigenvalues of $A$. Here, $\lambda_i > 0$. Define $\sqrt{D} = diag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n})$. Set $B = \sqrt{D}P^T$, then $B$ is invertible and $B^TB = A$.

Conversely, $X^TAX = X^TB^TBX = (BX)^T(BX) = \|BX\|^2$. Therefore, for $X \neq 0$, $X^TAX > 0$. □

Let $A \in M_n(\mathbb{R})$. The leading principal minor $D_k$ of $A$ of order $k$, $1 \leq k \leq n$, is the determinant of the matrix obtained from $A$ by deleting last $n-k$ rows and last $n-k$ columns of $A$.

**Proposition 6.** *Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Then*
*1. $A$ is positive definite if and only if $D_k > 0$ for $1 \leq k \leq n$.*
*2. $A$ is negative definite if and only if $(-1)^kD_k > 0$ for $1 \leq k \leq n$.*
*3. $A$ is positive semi-definite, then $D_k \geq 0$ for $1 \leq k \leq n$. Show that the converse need not be true.*
*4. $A$ is negative semi-definite, then $(-1)^kD_k \geq 0$ for $1 \leq k \leq n$. Show that the converse need not be true.*

*Proof.* The prove for this result has been omitted. To see that converse is not true in case of (3), take $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1/2 \end{pmatrix}$. Then $D_1 = 1$, $D_2 = \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$ and $D_3 = \det(A) = 0$. The matrix is symmetric and $D_k \geq 0$ for $k = 1, 2, 3$. But $X^T AX = -2$ for $X = (1, 1, -2)^T$. Therefore, $A$ is not positive semi-definite. □

**Exercise 1.** *Which of the following matrices is positive definite/negative definite/positive semi-definite/ negative semi-definite.*

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

**Singular-Value Decomposition**

We know that every matrix is not diagonalizable and diagonalizability can be discussed only for square matrices. Here we discuss a decomposition of an $m \times n$ matrix which coincide with a known decomposition of a positive semi-definite matrix.

Let $A \in M_{m\times n}$. Then a decomposition of the form

$$A = U\Sigma V^T,$$

where $U \in M_m(\mathbb{R})$ and $V \in M_n(\mathbb{R})$ are orthogonal, and $\Sigma$ is a rectangular diagonal matrix with non-negative real diagonal entries, is called Singular-Value Decomposition of $A$. The non-zero diagonal entries of $\Sigma$ are called singular values of $A$.

When $A$ is a positive semi-definite matrix, then SVD is nothing but $A = PDP^T$ for some orthogonal matrix $P$.

**Theorem 7.** *Let $A \in M_{m\times n}(\mathbb{R})$. Then $A$ has a singular value decomposition.*

**Proposition 8.** *Let $A \in M_{m\times n}(\mathbb{R})$. Then*
*1. $A^T A$ is positive semi-definite.*
*2. $AA^T$ is positive semi-definite.*
*3. If $m \geq n$, then $P^T(A^T A)P = D$ and $P'^T(AA^T)P' = D'$ for some orthogonal matrices $P \in M_n(\mathbb{R})$ and $P' \in M_m(\mathbb{R})$ with*

$$D' = \begin{pmatrix} D & 0_{m\times m-n} \\ 0_{m-n\times m} & 0_{m-n\times m-n} \end{pmatrix}.$$

*Proof.* Note that $A^T A$ and $AA^T$ are symmetric matrices. We claim that $X^T AX \geq 0$ for every $X \neq 0$. For $X \neq 0$, $X^T AA^T X = (A^T X)^T(A^T X) = \|A^T X\|^2 \geq 0$. Therefore, $AA^T$ is positive semi-definite. Similarly for $A^T A$. Since the $A^T A$ and $AA^T$ are symmetric, they are orthogonally diagonalizable. Therefore, $P^T(A^T A)P = D$ and $P'^T(AA^T)P = D'$ for some orthogonal matrices $P \in M_m(\mathbb{R})$ and $P' \in M_n(\mathbb{R})$. Recall that $p_{AA^T}x = x^{m-n}p_{A^T A}(x)$, where $p_{AA^T}(x)$ and $p_{A^T A}$ are the characteristic polynomial of $AA^T$ and $A^T A$ respectively. Hence, $D' = \begin{pmatrix} D & 0_{m\times m-n} \\ 0_{m-n\times m} & 0_{m-n\times m-n} \end{pmatrix}$. □

**Method to find SVD of $A$**

**Step 1:** Find $AA^T$, which is positive semi-definite matrix. Therefore, we can find an orthogonal matrix $U \in M_m(\mathbb{R})$ such that

$$U^T(AA^T)U = D.$$

Note that columns of $U$ are eigenvectors (orthonormal) of $AA^T$.

**Step 2:** Find $A^TA$, which is positive semi-definite matrix. We can find an orthogonal matrix $V \in M_n(\mathbb{R})$ such that

$$V^T(A^TA)V = D'.$$

Note that columns of $V$ are eigenvectors (orthonormal) of $A^TA$.

**Step 3:** Define a rectangular diagonal matrix $\Sigma \in M_{m\times n}$ such that $\Sigma_{ii} = \sqrt{\lambda_i}$ for $i = 1, 2, \ldots, \min(m, n)$, where $\lambda_i$ are the common eigenvalues of $A^TA$ and $AA^T$. Note that non-zero diagonal entries $\sigma_i$ are corresponding to non-zero eigenvalues of $A^TA$ or $AA^T$.

**Step 4:** Verify that $U\Sigma V^T = A$.

**Remark 9.** *Let $A \in M_{m\times n}(\mathbb{R})$ and rank$(A) = r$. Let $U\Sigma V^T$ be a singular value decomposition of $A$. Let $U_1, U_2, \ldots, U_m$ are columns of $U$ and $V_1, V_2, \ldots, V_n$ are columns of $V$. Then*

*1. $\{U_1, U_2, \ldots, U_r\}$ is an orthonormal basis of column space$(A)$.*

*2. $\{V_{r+1}, V_{r+2}, \ldots, V_n\}$ is an orthonormal basis of null space$(A)$.*

*3. $\{V_1, V_2, \ldots, V_r\}$ is an orthonormal basis of Column space of $(A^T)$ or row space of $A$.*

*4. $\{U_{r+1}, U_{r+2}, \ldots, U_n\}$ is an orthonormal basis of null space$(A^T)$.*

*Proof.* Note that $AV = U\Sigma \Rightarrow AV_j = \sigma_i U_j$ for $j = 1, 2, \ldots, r$ and $AV_j = 0$ for $j = r+1, \ldots, n$. Since nullity of $A$ is $n - r$ and $V_{r+1}, V_{r+2}, \ldots, V_n$ forms an orthonormal basis of $N(A)$. Since $\sigma_j > 0$ and $AV_j = \sigma_j U_j$, $U_j \in C(A)$ for $j = 1, 2, \ldots, r$. Thus $\{U_1, U_2, \ldots, U_r\}$ is an orthonormal basis of $C(A)$. Similarly, $A^TU = V\Sigma$ gives that first $r$ columns of $V$ forms a basis of the column space of $A^T$. □

**Example 10.** *Find SVD of* $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

**Solution:** $AA^T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $A^TA = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. Then $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Note that non-zero eigenvalue of $A^TA$ is 2 (as non-zero eigenvalue of $AA^T$ is 2) with eigenvectors $(0, 1, 0, 1)$ and $(1, 0, 1, 0)$ and the remaining eigenvalues of $A^TA$ are all zero. The eigenvectors corresponding to 0 are $(1, 0, -1, 0)$ and $(0, 1, 0, -1)$. Thus $V = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$. The rectangular diagonal matrix $\Sigma = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{pmatrix}$.

Therefore, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$.

**Remark:** After finding $U$, one can find columns of $V$ corresponding to non-zero eigenvalues by using the relation $V_i = \frac{1}{\sigma_i} A^T U_i$. The other columns of $V$ can be found by finding vectors orthogonal to $V_1, V_2$ and to each other.