**Lecture 21**
**Decomposition of a Matrix in Terms of Projections**

Here we discuss a special kind of linear maps (matrices), called projection and their properties. Further, we see that every diagonalizable matrix can be decomposed into projection matrices.

**Definition 1.** *Let $V$ be a vector space over $\mathbb{F}$. A linear map $E : V \to V$ is called a projection if $E^2 = E$. A matrix $M$ is called a projection matrix if $M^2 = M$, i.e., $M$ is idempotent.*

**Theorem 2.** *Let $E : V \to V$ be a projection. Let $R$ be the range of $E$ and $N$ be its null space. Then $V = R \oplus N$.*

**Proof:** It is easy to see that $R \cap N = \{0\}$. For $v \in V$, let $v = v - Ev + Ev \in N + R$.

**Theorem 3.** *Let $R$ and $N$ be subspaces of a vector space $V$ such that $V = R \oplus N$. Then there is a projection map $E$ on $V$ such that the range of $E$ is $R$ and the null space of $E$ is $N$.*

**Proof:** Define $E : V \to V$ as $E(r + n) = r$.

**Definition 4.** *A vector space $V$ is said to be a direct sum of $k$ subspaces $W_1, W_2, \ldots, W_k$ if $V = W_1 + W_2 + \cdots + W_k$ and $W_i \cap (W_1 + W_2 + \cdots + W_{i-1} + W_{i+1} + \cdots + W_k) = \{0\}$ for each $i$.*

**Theorem 5.** *If $V = W_1 \oplus W_2 \ldots \oplus W_k$, then there exist $k$ linear maps $E_1, \ldots, E_k$ on $V$ such that:*

1. *Each $E_i$ is projection,*
2. *$E_iE_j = 0$ for all $i \neq j$,*
3. *$E_1 + \ldots + E_k = I$,*
4. *the range of $E_i$ is $W_i$.*

*Proof.* Let $v \in V$. Then $v = w_1 + w_2 + \cdots + w_k$, where $w_i \in W_i$. Define $E_i : V \to V$ as $E_i(v) = E_i(w_1 + \ldots + w_k) = w_i$ for all $i$. Then $E_i$ is linear with $E_i^2(v) = v$ for all $v \in V$. Also $E_iE_j = 0$ for all $i \neq j$ and $E_1 + \ldots + E_k = I$. By definition of $E_i$, range of $E_i$ is $W_i$. $\square$

**Lemma 6.** *Let $A \in M_n(\mathbb{F})$. The matrix $A$ is diagonalizable if and only if $\mathbb{F}^n = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$, where $\lambda_i \in \mathbb{F}$ and $\lambda_i \neq \lambda_j$ for $i \neq j$ and $E_{\lambda_i}$ is the eigenspace of $\lambda_i$.*

*Proof.* Let $A$ be diagonalizable. Recall that if $B_i$ is a basis of the eigenspace $E_{\lambda_i}$, then $\cup_{i=1}^k B_i$ is a basis of $V = \mathbb{F}^n$. Thus $V = E_{\lambda_1} + \cdots + E_{\lambda_k}$. Let $v \in E_{\lambda_i} \cap (E_{\lambda_1} + E_{\lambda_2} + \cdots + E_{\lambda_{i-1}} + E_{\lambda_{i+1}} + \cdots + E_{\lambda_k})$. Then $Av = \lambda_i v$ and $v = v_1 + v_2 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_k$, where $v_j \in E_{\lambda_j}$ and $j \neq i$. Then $Av = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{i-1} v_{i-1} + \lambda_{i+1} v_{i+1} + \cdots + \lambda_k v_k$ so that $(\lambda_i - \lambda_1)v_1 + \cdots + (\lambda_i - \lambda_{i-1})v_{i-1} + (\lambda_i - \lambda_{i+1})v_{i+1} + \cdots + (\lambda_i - \lambda_k)v_k = 0$. If $v$ is non-zero, not all $v_i$ are zero. Note that if $v_j \neq 0$, it is an eigenvector corresponding to $\lambda_j$, but eigenvectors corresponding to distinct eigenvalues are independent, hence $\lambda_i = \lambda_j$ for some $j \neq i$, which is a contradiction. $\square$

**Theorem 7.** *Let $A$ be a diagonalizable matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then $A$ can be decomposed as a linear sum of idempotent (projection) matrices $E_1, \ldots, E_k$ given by $A = \lambda_1 E_1 + \ldots + \lambda_k E_k$.*

**Proof:** The matrix $A$ is diagonalizable so that the minimal polynomial of $A$ is $(x-\lambda_1)\dots(x-\lambda_k)$. Define

$$E_j = \frac{(A-\lambda_1 I)\dots(A-\lambda_{j-1}I)(A-\lambda_{j+1}I)\dots(A-\lambda_k I))}{(\lambda_j-\lambda_1)\dots(\lambda_j-\lambda_{j-1})(\lambda_j-\lambda_{j+1})\dots(\lambda_j-\lambda_k)}.$$

Let $v \in V$, then $v = v_1 + v_2 + \cdots + v_k$, where $v_i \in E_{\lambda_i}$. Let $v_i \in E_{\lambda_i}$, then $E_j(v_i) = 0$ if $i \neq j$ and $E_j(v_j) = v_j$ so that $E_j(v) = E_j(v_1 + v_2 + \cdots + v_k) = E_j(v_1) + E_j(v_2) + \cdots + E_j(v_k) = v_j$. Thus $E_j$ is a projection matrix. One can see that (i) $E_i^2 = E_i$, (ii) $E_iE_j = 0$ and $I = E_1 + \ldots + E_k$ (left to the reader to verify). Now $I = E_1 + E_2 + \cdots + E_k$ so that $A = AE_1 + AE_2 + \cdots + AE_k$. Then $Av = A(v_1 + v_2 + \cdots + v_k) = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_k v_k = \lambda_1 E_1(v) + \lambda_2 E_2(v) + \cdots + \lambda_k E_k(v)$ for all $v \in V$. Therefore, $A = \lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_k E_k$.

**Example:** Check the diagonalizability of the given matrix $\begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. If diagonalizable, write the matrix as linear sum of projection matrices.

**Solution:** The characteristic polynomial $p(x) = (x-1)(x-2)^2$. Let $\lambda_1 = 1$ and $\lambda_2 = 2$. Then $GM(1) =$ and eigenvectors corresponding to 2 are $v_2 = (2,1,0)$ and $(2,0,1)$ so that $GM(2) = 2$. Hence, the matrix is diagonalizable. Then as per the above theory, $E_1 = (2I - A)$ and $E_2 = (A - I)$ and hence, $A = 1(2I - A) + 2(A - I)$. Verify yourself that $E_i^2 = E_i$ for $i = 1, 2$.