Lecture 20
Spectral Theorem

Definition 1 (Orthogonal Matrix)**.** *A real square matrix is called orthogonal if* $AA^T = I = A^T A$.

Definition 2 (Unitary Matrix)**.** *A complex square matrix is called unitary if* $AA^* = I = A^*A$*, where* A^* *is the conjugate transpose of* A*, that is,* $A^* = \overline{A}^T$.

Theorem 3. Let A be a unitary (real orthogonal) matrix. Then
(i) rows of A forms an orthonormal set;
(ii) columns of A forms an orthonormal set.

Remark 4. *1. P is orthogonal if and only if P^T is orthogonal.*
2. P is unitary if and only if P^* is unitary.
3. An orthogonal matrix (unitary) is invertible and its inverse is orthogonal (unitary).
4. Product of two orthogonal (unitary) matrices is orthogonal (unitary).

Theorem 5. The eigenvalues of a unitary matrix (an orthogonal matrix) has absolute value 1.

Proof: Let λ be an eigenvalue of a unitary matrix A. Then there exists a non-zero vector X such that $AX = \lambda X$. Thus, $(AX)^* = \bar{\lambda} X^* \Rightarrow (AX)^*(AX) = \bar{\lambda} X^*(\lambda X) \Rightarrow X^*A^*AX = \lambda\bar{\lambda} X^*X$. But $A^*A = I$, $(1 - |\lambda|^2)X^*X = 0$, *i.e.*, $|\lambda| = 1$.

Definition 6. A complex square matrix A is called a Hermitian matrix if $A = A^*$, where A^* is the conjugate transpose of A, that is, $A^* = \overline{A}^T$. A complex square matrix is called skew-Hermitian if $A = -A^*$.

Theorem 7. 1. The eigenvalues of a Hermitian matrix (real symmetric matrix) are real.
2. The eigenvalues of a skew-Hermitian matrix (real skew-symmetric matrix) are either purely imaginary or zero.

Proof: Let λ be an eigenvalue of a Hermitian matrix A. Then there exists a non-zero vector $X \in \mathbb{C}^n$ such that $AX = \lambda X$, multiplying both side by X^*, we get $X^*AX = \lambda X^*X$. Taking conjugate transpose both sides, we get $(X^*AX)^* = (\lambda X^*X)^* \Rightarrow X^*AX = \bar{\lambda} X^*X$. Thus we see that $\lambda X^*X = \bar{\lambda} X^*X$. Since $X \neq 0$, $X^*X = ||X||^2 \neq 0$ so that $\lambda = \bar{\lambda}$. For skew-Hermitian matrix, proceed in a similar way.

Theorem 8. Let A be a real symmetric matrix. Then eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

Proof: Let $\lambda_1 \neq \lambda_2$ be two eigenvalues of A and v_1 and v_2 be corresponding eigenvectors respectively. Then $Av_1 = \lambda_1 v_1 \Rightarrow v_1^T A^T = \lambda_1 v_1^T \Rightarrow v_1^T A^T v_2 = \lambda_1 v_1^T v_2$. Also $(Av_1)^T v_2 = v_1^T A^T v_2 = v_1^T A v_2 = \lambda_2 v_1^T v_2$. Hence, $(\lambda_1 - \lambda_2) v_1^T v_2 = 0$, and $\lambda_1 \neq \lambda_2$ so that $v_1^T v_2 = 0 = \langle v_1, v_2 \rangle \Rightarrow v_1 \perp v_2$.

Theorem 9. [**Spectral Theorem for a real symmetric matrix**] Let A be a real symmetric matrix. Then there exists an orthogonal matrix P such that $P^T AP = D$, where D is a diagonal matrix. In other words, a real symmetric matrix is orthogonally diagonalizable.

Proof: The proof is by induction on order of the matrix. The result holds for $n = 1$. Suppose the result holds for $(n-1) \times (n-1)$ symmetric matrix. Let A be a symmetric matrix of order $n \times n$. Note that A has real eigenvalues. Let $\lambda \in \mathbb{R}$ be one of the eigenvalue and $0 \neq X \in \mathbb{R}^n$ be a corresponding eigenvector with norm 1, then $AX = \lambda X$. Construct an orthonormal basis (by Gram-Schmidt process) $B = \{v_1, v_2, v_3, \dots, v_n\}$, where $v_1 = X$ and $v_i \in \mathbb{R}^n$. Construct a matrix P such that the i-th column of P is v_i. Then P is an orthogonal matrix.

Note that the matrix $P^{-1}AP$ is symmetric and the first column of $P^{-1}AP$ is given by $P^{-1}AP(e_1)$, thus $P^{-1}A(Pe_1) = P^{-1}AX = P^{-1}\lambda X = \lambda e_1$. Therefore, the matrix can be represented as $P^{-1}AP = \begin{bmatrix} \lambda & 0 \\ 0 & C \end{bmatrix}$, where C is a symmetric matrix of order $(n-1) \times (n-1)$. Hence, by induction hypothesis, C is similar to a diagonal matrix, say D, *i.e.*, there is an orthogonal matrix Q such that $Q^{-1}CQ = Q^TCQ = D$. Let $R = P\begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$. We claim that R is orthogonal and R^TAR is diagonal.

$$R^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & Q^T \end{bmatrix} P^T = R^T, \text{ and}$$

$$R^TAR = \begin{bmatrix} 1 & 0 \\ 0 & Q^T \end{bmatrix} P^TAP \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q^T \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & Q^TCQ \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix}.$$

Thus R is an orthogonal matrix such that R^TAR is diagonal. Therefore, A is orthogonally diagonalizable.
□

Theorem 10. Converse of the above theorem is also true, *i.e.*, if $A \in M_n(\mathbb{R})$ is orthogonally diagonalizable, then A is symmetric.

Proof: Let A be a matrix which is orthogonally diagonalizable. Then there is an orthogonal matrix P s.t. $P^{-1}AP = P^TAP = D$, equivalentely, $A = PDP^{-1} = PDP^T$. This shows that $A^T = A$. Hence proved.

Example: Find an orthogonal matrix P and a diagonal matrix D such that $P^TAP = D$, where

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

The characteristic polynomial is $(x+1)^2(x-5)$. The eigenvalues are $5, -1, -1$. An eigenvector corresponding to $\lambda = 5$ is $v_1 = (1,1,1)$. The two independent eigenvectors corresponding to $\lambda = -1$ are $v_2 = (-1,0,1)$ and $v_3 = (-1,1,0)$. Thus, $B = \{v_1, v_2, v_3\}$ forms a basis of $\mathbb{R}^3$. To find an orthonormal basis, we apply Gram-Schmidt process on B. Thus

$w_1 = v_1$, $||w_1|| = \sqrt{3}$,
$w_2 = v_2$ (eigen vectors corresponding to distinct eigen values are orthogonal), $||w_2|| = \sqrt{2}$,
$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle w_1}{||w_1||^2} - \frac{\langle v_3, w_2 \rangle w_2}{||w_2||^2} = (-1,1,0) - 0(1,1,1) - \frac{1(-1,0,1)}{2} = (-\frac{1}{2}, 1, -\frac{1}{2})$, $||w_3|| = \frac{\sqrt{6}}{2}$

Thus, $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{6}}{3} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Verify yourself that $P^TAP = D$.