Lecture 17
Inner Product Space

Let $V = \mathbb{R}^2$ and $P = (x_1, x_2)$ and $Q = (y_1, y_2)$ be two vectors in V. The dot product of P and Q is defined as $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1 + x_2y_2$. Then the length of P, $||P|| = \sqrt{(x_1, x_2) \cdot (x_1, x_2)}$, distance between P and Q is $d(p, q) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \sqrt{(x_1 - y_1, x_2 - y_2).(x_1 - y_1, x_2 - y_2)}$ and the angle (θ) between P and Q is defined as $cos\theta = \frac{P.Q}{||P||||Q||}$.

Observe that the above dot product satisfies the following properties:

1. $(x \cdot x) \geq 0$ and $(x \cdot x) = 0$ if and only if $x = 0$;
2. $(x \cdot y) = (y \cdot x)$, $\forall x, y \in \mathbb{R}^n$;
3. $((\alpha x) \cdot y) = \alpha(x \cdot y)$, $\forall \alpha \in \mathbb{R}$;
4. $((x + y) \cdot z) = (x \cdot z) + (y \cdot z)$.

In an arbitrary vector space, we define a function which satisfies the above four conditions, we call this function **inner product**, with the help of this function we can define the geometric concepts such as length of a vector, distance between two vectors and angle between the vectors.

Definition 1. Let V be a vector space over $\mathbb{F}$. A function $\langle\, , \rangle : V \times V \longrightarrow \mathbb{F}$ is called an inner product on V if it satisfies the following properties.
1. $\langle x, x\rangle \geq 0$ $\forall x \in V$ and $\langle x, x\rangle = 0$ if and only if $x = 0$;
2. $\langle x, y\rangle = \overline{\langle y, x\rangle}$, $\forall x, y \in V$;
3. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$, $\forall \alpha \in \mathbb{F}$ and $\forall x, y, z \in V$.

A vector space $V(\mathbb{F})$ together with an inner product $\langle\, , \rangle$ is called an inner product space and denoted by $(V, \langle\, , \rangle)$.

Example 2. *1. Let $V = \mathbb{R}^n$ over $\mathbb{R}$ with $\langle x, y\rangle = x \cdot y$, that is, $\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) = x_1y_1 + x_2y_2 + \cdots + x_ny_n$.*
2. Let $V = \mathbb{C}^n$ over $\mathbb{C}$. Define $\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) = x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n}$.
3. Let $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$ and $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ such that $a, c > 0$ and $ac - b^2 > 0$. Define $\langle x, y\rangle = y^T Ax$.
4. Let $V = C[a, b]$, $\mathbb{F} = \mathbb{R}$. Define $\langle f(x), g(x)\rangle = \int_a^b f(x)\overline{g(x)}dx$.
5. Let $V = M_n(\mathbb{R})$, $\mathbb{F} = \mathbb{R}$. Then for $A, B \in V$, define $\langle A, B\rangle = trace(AB^T)$.

Proposition 3. *Every finite dimensional vector space is an inner product space.*

Proof. Let $B = \{v_1, \ldots v_n\}$ be an ordered basis of $V(\mathbb{F})$. Then for $u, v \in V$, define $\langle u, v\rangle = \alpha_1\overline{\beta_1} + \ldots + \alpha_n\overline{\beta_n}$, where $(\alpha_1, \ldots, \alpha_n)^T = [u]_B$ and $(\beta_1, \ldots, \beta_n)^T = [v]_B$. $\square$

Note that $\langle v, v\rangle > 0$ for non-zero $v \in V$. This leads us to define the concept of length of a vector in an inner product space.

Definition 4. The length of a vector v (norm of a vector v) is defined as $||v|| = \sqrt{\langle v, v\rangle}$.

Theorem 5 (Cauchy-Schwartz Inequality). *Let V be an inner product space. Then $|\langle v, u\rangle| \leq ||v||\,||u||$, $\forall u, v \in V$. The equality holds if and only if the set $\{u, v\}$ is linearly dependent.*

Proof: Clearly, the result is true for $u = 0$. Suppose $u \neq 0$. Let $w = v - \frac{\langle v,u\rangle}{||u||^2}u$. Then $w \in V$. By the property $\langle w, w\rangle \geq 0$, we get $||v||^2 - \frac{|\langle v,u\rangle|^2}{||u||^2} \geq 0$. Therefore, $|\langle v, u\rangle| \leq ||v||\,||u||$.

For equality, if $u = 0$ then the set $\{0, v\}$ is L.D.. If $u \neq 0$ then from the above we have $v = \frac{\langle v,u\rangle}{||u||^2}u$. Conversely, let u, v are L.D. then $u = \alpha v$ for some $\alpha \in \mathbb{F}$. Then $|\langle u, v\rangle| = |\langle \alpha v, v\rangle| = |\alpha|||v||^2 = ||u||\,||v||$.

Proposition 6. *Let $(V(\mathbb{F}), \langle\, ,\rangle)$ be an inner product space. Then*

1. $||u + v|| \leq ||v|| + ||u||$, $\forall u, v \in V$. (Triangle inequality)

2. $||u + v||^2 + ||u - v||^2 = 2(||v|| + ||u||)^2$ $\forall u, v \in V$. (Parallelogram law)

Proof: By definition, $||u+v||^2 = \langle u+v, u+v\rangle = ||v||^2 + \langle u, v\rangle + \overline{\langle u, v\rangle} + ||u||^2 = ||v||^2 + 2Re(\langle u, v\rangle) + ||u||^2 \leq ||v||^2 + 2|\langle u, v\rangle| + ||u||^2 = (||u|| + ||v||)^2$. Prove the second statement yourself.

Definition 7. *Let u and v be vectors in an inner product space $(V, \langle\, ,\rangle)$. Then u and v are* **orthogonal** *if $\langle u, v\rangle = 0$. A set S of an inner product space is called an* **orthogonal set** *of vectors if $\langle u, v\rangle = 0$ for all $u, v \in S$ and $u \neq v$. An* **orthonormal set** *is an orthogonal set S with the additional property that $||u|| = 1$ for every $u \in S$.*

Proposition 8. *An orthogonal set of non-zero vectors is linearly independent.*

Proof: Let S be an orthogonal set (finite or infinite) of non-zero vectors in a given inner product space. Suppose $v_I, v_2, \ldots, v_m$ are distinct vectors in S and take $w = \alpha_1 v_1 + \cdots + \alpha_m v_m$. Then $\langle w, v_i\rangle = \langle \alpha_1 v_1 + \cdots + \alpha_m v_m, v_i\rangle = \alpha_1\langle v_1, v_i\rangle + \alpha_2\langle v_2, v_i\rangle + \cdots + \alpha_m\langle v_m, v_i\rangle = \alpha_i\langle v_i, v_i\rangle$. Note that $v_i \neq 0$ so that $\langle v_i, v_i\rangle \neq 0$. If $w = 0$, then $\alpha_i = 0$ for each i. Therefore, S is linearly independent.

Gram-Schmidt orthogonalization process

Theorem 9. *Let $(V, \langle\,,\rangle)$ be an inner product space and $S = \{v_1, v_2, \ldots, v_n\}$ be a linearly independent set of vectors in V. Then we get an orthogonal set $\{w_1, w_2, \ldots, w_n\}$ in V such that*

$$L(\{v_1, v_2, \ldots, v_n\}) = L(\{w_1, w_2, \ldots, w_n\}).$$

Proof. $w_1 = v_1$, then $L(\{w_1\}) = L(\{v_1\})$;
$w_2 = v_2 - \frac{\angle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$, then $\langle w_2, w_1\rangle = 0$ with $L(\{w_1, w_2\}) = L(\{v_1, v_2\})$;
$w_3 = v_3 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1$, then $\langle w_3, w_1\rangle = 0$, and $\langle w_3, w_2\rangle = 0$ with $L(\{w_1, w_2, w_3\}) = L(\{v_1, v_2, v_3\})$;
Inductively,
$w_n = v_n - \frac{\langle v_n, w_{n-1}\rangle}{\langle w_{n-1}, w_{n-1}\rangle} w_{n-1} - \frac{\langle v_n, w_{n-2}\rangle}{\langle w_{n-2}, w_{n-2}\rangle} w_{n-2} - \cdots - \frac{\langle v_n, w_1\rangle}{\langle w_1, w_1\rangle} w_1$, then $\langle w_n, w_i\rangle = 0$ for $i \neq n$ with $L(\{v_1, v_2, \ldots, v_n\}) = L(\{w_1, w_2, \ldots, w_n\})$. □

Remark 10. *1. The method by means of which orthogonal vectors $w_1, \ldots, w_n$ are obtained is known as the* **Gram-Schmidt orthogonalization process**.
2. Every finite-dimensional inner product space has an orthonormal basis.
3.Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for an inner product space V. Then for any $w \in V$, $w = \langle w, v_1\rangle v_1 + \cdots + \langle w, v_n\rangle v_n$.

Example 11. *Find an orthogonal basis of $\mathbb{R}^2$ with the inner product given by $\langle (x_1, y_1), (x_2, y_2)\rangle = x_1x_2 + 2x_1y_2 + 2x_2y_1 + 5y_1y_2$.*

Solution: We know that $\{e_1, e_2\}$ is a basis of $\mathbb{R}^2$. Since $\langle e_1, e_2\rangle = 2 \neq 0$, the standard basis is not an orthogonal basis under the defined inner product. To get an orthogonal basis we use Gram-Schmidt process: $w_1 = e_1$ and $w_2 = e_2 - \langle e_2, e_1\rangle \frac{e_1}{||e_1||^2}$ and $||e_1||^2 = \langle e_1, e_1\rangle = 1$ so that $w_2 = e_2 - 2e_1$. Thus $\{e_1, e_2 - 2e_1\}$ is an orthogonal basis.