Lecture 16

(Cayley Hamilton Theorem, minimal polynomial & Diagonalizability)

Theorem 1. Cayley-Hamilton Theorem: Every square matrix satisfies its characteristic equation, that is, if $f(x)$ is the characteristic polynomial of a square matrix A, then $f(A) = 0$.

Example 2. *Let* $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. *Find inverse of A using Cayley-Hamilton theorem.*

Solution: The characteristic polynomial of A is $f(x) = x^3 - 2x^2 + 1$. The constant term of $f(x) = 1 = \det(A)$, the matrix A is invertible. By Cayley-Hamilton Theorem $f(A) = 0$. Therefore $A^3 - 2A^2 + I = 0 \Rightarrow A^{-1} = -A^2 + 2A \Rightarrow -\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

Definition 3. A polynomial $m(x)$ is said to be the minimal polynomial of A if

(*i*) $m(A) = 0$;

(*ii*) $m(x)$ is a monic polynomial (the coefficient of the highest degree term is 1);

(*iii*) if a polynomial $g(x)$ is such that $g(A) = 0$, then $m(x)$ divides $g(x)$.

Remark 4. 1.The minimal polynomial of a matrix is unique.
2. The minimal polynomial divides its characteristic polynomial.

Theorem 5. The minimal polynomial and the characteristic polynomial have the same roots.

Proof: Let $f(x)$ and $m(x)$ be the characteristic and minimal polynomial of a matrix respectively. Then $f(x) = g(x)m(x)$. If α is a root of $m(x)$, then it is also a root of $f(x)$. Conversely, if α is a root of $f(x)$, then α is an eigenvalue of the matrix. Therefore, there is a non-zero eigenvector v such that $Av = \alpha v$, this implies $m(A)v = m(\alpha)v$, *i.e.*, $m(\alpha)v = 0$, and $v \neq 0$ so that $m(\alpha) = 0$. □

Theorem 6. Similar matrices have the same minimal polynomials.

Proof: Let A and B be two similar matrices. Then $A = P^{-1}BP$ for some invertible matrix P. Let $m_1(x) = a_0 + a_1x + \ldots + x^n$ and $m_2(x) = b_0 + b_1x + \ldots + x^l$ be the respective minimal polynomials of A and B. Then $m_2(A) = 0$, which implies $m_1(x)|m_2(x)$. Similarly $m_1(B) = 0$, which implies $m_2(x)|m_1(x)$.
□

Theorem 7. Let $A \in M_n(\mathbb{F})$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{F}$ be all eigenvalues of A, where $\lambda_i \neq \lambda_j$ for $i \neq j$. The matrix A is diagonalizable if and only if its minimal polynomial is a product of distinct linear polynomials, that is, $m(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_k)$, where λ_i's are distinct elements of $\mathbb{F}$.

Example 8. *A matrices $A \in M_n(\mathbb{R})$ such that $A^2 - 3A + 2I = 0$ is diagonalizable.*

Solution: *Take $g(x) = x^2 - 3x + 2$, then $g(A) = 0$. Note that $g(x) = (x-1)(x-2)$ and the minimal polynomial $m(x)$ of A divides $g(x)$. Therefore, either $m(x) = (x-1)$ or $m(x) = (x-2)$ or $m(x) = (x-1)(x-2)$. In either of the case, the minimal polynomial is a product of distinct linear polynomials, hence diagonalizable.*