Lecture 14 (Eigenvalue & Eigenvector)

Definition 1. Let V be a vector space over $\mathbb{F}$ and $T : V \to V$ be a linear transformation. Then
1. a scalar $\lambda \in \mathbb{F}$ is said to be an **eigenvalue** or **characteristic value** of T if there exists a non-zero vector $v \in V$ such that $Tv = \lambda v$.
2. a non-zero vector v satisfying $Tv = \lambda v$ is called **eigenvector** or **characteristic vector of T associated to the eigenvalue λ**.
3. The set $E_\lambda = \{v \in V : Tv = \lambda v\}$ is called the **eigenspace of T associated to the eigenvalue λ**.

Example 2. *Let V be a non-zero vector space over $\mathbb{F}$.*
1. If T is the zero operator, zero is the only eigenvalue of T.
2. For identity operator, one is the only eigenvalue.
3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (0, x)$. Then $T(x, y) = \lambda(x, y) \Leftrightarrow (0, x) = (\lambda x, \lambda y) \Leftrightarrow (\lambda x = 0, y = \lambda y \Leftrightarrow \lambda = 0, x = 0, y \neq 0$. Thus, 0 is the eigenvalue of T and $(0, 1)$ is an eigenvector corresponding to 0.
4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (y, -x)$. Then $T(x, y) = \lambda(x, y) \Leftrightarrow (y, -x) = (\lambda x, \lambda y) \Leftrightarrow (\lambda^2 + 1)x = 0 \Leftrightarrow \lambda = \pm i, x \neq 0$. Thus, T has no real eigenvalue.
5. Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ given by $T(x, y) = (y, -x)$. Then $T(x, y) = \lambda(x, y) \Leftrightarrow (y, -x) = (\lambda x, \lambda y) \Leftrightarrow (\lambda^2 + 1)x = 0 \Leftrightarrow \lambda = \pm i, x \neq 0$. Thus, T has two complex eigenvalues $\pm i$ and $(1, i)$ is an eigenvector corresponding to i and $(1, -i)$ is an eigenvector corresponding to $-i$.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (2x + 3y, 3x + 2y)$. To find $\lambda \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$ such that $(2x + 3y, 3x + 2y) = \lambda(x, y)$ or $(2 - \lambda)x + 3y = 0, 3x + (2 - \lambda)y = 0$. The system of linear equations has a non-zero solution if and only if the determinant of the coefficient matrix, $\det \begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = 0$ *or $\lambda = -1, 5$. When $\lambda = 1$, $3x + 3y = 0$ so that $(1, -1)$ is an eigenvector ($(-a, a)$ are eigenvectors of corresponding to eigenvalue -1 for every $a \neq 0$). For $\lambda = 5$, $3x - 3y = 0$ so that $(1, 1)$ is an eigenvector (in fact, (a, a) is an eigenvector corresponding to eigenvalue 5 for $a \neq 0$).*

Theorem 3. *Let T be a linear operator on a finite-dimensional vector space $V(\mathbb{F})$ and $\lambda \in \mathbb{F}$. The following statements are equivalent.*
1. λ is an eigenvalue of T.
2. The operator $T - \lambda I$ is singular (not invertible).
3. $\det[(T - \lambda I)]_B = 0$, where B is an ordered basis of V.

Proof. A linear transformation T is singular if and only if $ker(T) \neq \{0\}$. Thus, (1) $\iff$ (2). if $V(\mathbb{F})$ is finite-dimensional, then the eigenvalues and eigenvectors of T can be determined by its matrix representation $[T]_B$ with respect to a basis B. A scalar λ is an eigenvalue of $T \Leftrightarrow Tv = \lambda v \Leftrightarrow [T]_B[v]_B = \lambda[v]_B \Leftrightarrow ([T]_B - \lambda I)[v]_B = 0$ for non zero v. Thus, (3) $\Leftrightarrow$ (1). $\square$

Definition 4. Let $A \in M_n(\mathbb{F})$. A scalar $\lambda \in \mathbb{F}$ is said to be an **eigenvalue of** A if there exists a non-zero vector $x \in \mathbb{F}^n$ such that $Ax = \lambda x$. Such a non-zero vector x is called an **eigenvector of** A **associated to the eigenvalue** λ.

Let $A \in M_n(\mathbb{F})$. Observe, $det(xI - A)$ is an n degree polynomial in x over $\mathbb{F}$. A scalar λ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda I) = 0$ or $\det(\lambda I - A) = 0$.

Definition 5. Let $A \in M_n(\mathbb{F})$. Then the polynomial $f(x) = \det(xI - A)$ is called the **characteristic polynomial** of A. The equation $\det(xI - A) = 0$ is called the **characteristic equation** of A.

Theorem 6. A scalar $\lambda \in \mathbb{C}$ is an eigenvalue if and only if λ is a root of the characteristic polynomial of A.

Example 7. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. The characteristic polynomial of A is $\det \begin{pmatrix} x-1 & -1 & 0 \\ 0 & x-1 & -1 \\ -1 & 0 & x-1 \end{pmatrix}$, that is, $x^3 - 3x^2 + 3x - 2 = (x-2)(x^2 - x + 1)$. Thus, the roots are $\lambda = 2, \frac{1 \pm \sqrt{3}i}{2}$. If $\mathbb{F} = \mathbb{R}$, the only eigenvalue of A is 2 and if $\mathbb{F} = \mathbb{C}$, the eigenvalues are $2, \frac{1 \pm \sqrt{3}i}{2}$. We leave it to the reader to find the corresponding eigenvectors over the field $\mathbb{C}$. In this example, we see that a real matrix over $\mathbb{C}$ may have complex eigenvalues.

Example 8. Consider a matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The characteristic polynomial is $x^2 + 1$ and the roots are $\pm i$. Thus, A has no eigenvalue over $\mathbb{R}$ and two eigenvalues over $\mathbb{C}$. Note that, the existence of eigenvalue depends on the field.

Properties of eigenvalue and eigenvector

1. Let $A \in M_n(\mathbb{C})$. Then the sum of eigenvalues is equal to the trace of the matrix and the product of eigenvalues is equal to the determinant of the matrix.

   Let
   $$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

   Then the characteristic polynomial of A is $f(\lambda) = |\lambda I - A| = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n$ with roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $\lambda_1 + \lambda_2 + \ldots + \lambda_n = -\frac{a_1}{a_0}$ and $\lambda_1\lambda_2 \ldots \lambda_n = (-1)^n \frac{a_n}{a_0}$.

Note that $a_0 = 1$, $f(0) = a_n = |-A| = (-1)^n|A|$ and $a_1 = -(a_{11} + a_{22} + \ldots + a_{nn})$. Therefore, $\lambda_1 + \lambda_2 + \ldots + \lambda_n = -\frac{a_1}{a_0} = (a_{11} + a_{22} + \ldots + a_{nn}) = trace(A)$ and $\lambda_1\lambda_2 \ldots \lambda_n = (-1)^n \frac{a_n}{a_0} = |A| = det(A)$.

2. If A is a non-singular matrix and λ is any eigenvalue of A, then λ^{-1} is an eigenvalue of A^{-1}.

   Let λ be an eigenvalue of A, then there exists $0 \neq x \in \mathbb{F}^n$ such that $Ax = \lambda x \Leftrightarrow A^{-1}x = \frac{1}{\lambda}x$.

3. A and and A^T have the same eigenvalues.

   It is enough to show that A and A^T have the same characteristic polynomials. The characteristic polynomial of A is $|\lambda I - A| = |(\lambda I - A)^T| = |\lambda I - A^T|$=characteristic polynomial of A^T.

4. Similar matrices have the same eigenvalues (or characteristic equations).

   Let A and B are two matrices which are similar then there exists an invertible matrix P such that $A = P^{-1}BP$. Then characteristic polynomial of A is $|\lambda I - A| = |\lambda I - P^{-1}BP| = |P^{-1}(\lambda I - B)P| = |\lambda I - B|$.

5. If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k for a positive integer k.

6. Let $\mu \in \mathbb{F}$ and $A \in M_n(\mathbb{F})$. Then $\lambda \in \mathbb{F}$ is an eigenvalue of A if and only if $\lambda \pm \mu$ is eigenvalue of $A \pm \mu I$.