Lecture 11

Rank-Nullity theorem & Vector Space Isomorphism

Theorem 1. Rank-Nullity Theorem: Let V and W be vector spaces over the field $\mathbb{F}$ and let $T : V \to W$ be a linear map. If V is finite dimensional then, $nullity(T) + rank(T) = \dim(V)$.

Proof: Since $Ker(T)$ is a subspace of V, its dimension is finite, say n. Let $B = \{v_1, \ldots, v_n\}$ be a basis for $Ker(T)$. Then B can be enlarged to form a basis for V. Let $B' = \{v_1, \ldots, v_n, v_{n+1}, \ldots, v_m\}$ be a basis for V. Now claim that the set $S = \{T(v_{n+1}), \ldots, T(v_m)\}$ forms a basis for $Range(T)$. Let $v \in V$. Then $v = \alpha_1 v_1 + \ldots + \alpha_m v_m$, this implies $T(v) = \alpha_{n+1}T(v_{n+1}) + \ldots + \alpha_m T(v_m)$. Thus $L(S) = Range(T)$. To show that S is linearly independent, assume that $\alpha_{n+1}T(v_{n+1}) + \ldots + \alpha_m T(v_m) = 0$. Then $T(\alpha_{n+1}v_{n+1} + \ldots + \alpha_m v_m) = 0$ so that $\alpha_{n+1}v_{n+1} + \ldots + \alpha_m v_m \in Ker(T)$. Therefore, $\alpha_{n+1}v_{n+1} + \ldots + \alpha_m v_m = \beta_1 v_1 + \ldots + \beta_n v_n$ or $\sum_{i=1}^{n} \beta_i v_i + \sum_{i=n+1}^{m} \alpha_i v_i = 0$. But B' is a basis for V. Therefore, $\alpha_i = 0$ and hence, S is linearly independent. □

Recall that a function $f : X \to Y$ is invertible if there exits a function $g : Y \to X$ such that $f \circ g = I_Y$ and $g \circ f = I_X$. Furthermore, a function$f$ is invertible if and only if it is one-one and onto, and the inverse function g is given by $g(y) = f^{-1}(y)$.

Theorem 2. *Let $T : V \to W$ be a linear map. If T is invertible, then the inverse map T^{-1} is linear.*

Proof: Suppose $T : V \longrightarrow W$ is invertible. Then T is one-one and onto. Let T^{-1} denote the inverse of T. We want to show that $T^{-1}(\alpha w_1 + \beta w_2) = \alpha T^{-1}(w_1) + \beta T^{-1}(w_2)$. Let $T^{-1}(w_1) = v_1$ and $T^{-1}(w_2) = v_2$. Then $T(\alpha v_1 + \beta v_2) = \alpha w_1 + \beta w_2$. Since T is one-one, $T^{-1}(\alpha w_1 + \beta w_2) = \alpha v_1 + \beta v_2 = \alpha T^{-1}(w_1) + \beta T^{-1}(w_2)$.

Definition 3. A linear map $T : V \to W$ is said to be non-singular if Ker(T) = $\{0\}$.

Theorem 4. A linear map $T : V \to W$ is non-singular if and only if T is one-one.

Proof: Let T is non-singular. If $T(x) = T(y)$, then $T(x - y) = 0$. This implies $x - y \in Ker(T) = \{0\}$. So $x = y$. Conversely, let $x \in Ker(T)$. Then $T(x) = 0 = T(0)$, as T is one one. So $x = 0$. □

Theorem 5. Let V and W be finite-dimensional vector spaces over the field $\mathbb{F}$ such that $\dim V = \dim W$. If T is a linear transformation from V to W, the following are equivalent:
(i) T is invertible.
(ii) T is non-singular.
(iii) T is onto, that is, the range of T is W.

Definition 6. Let V and W be vector spaces over the field $\mathbb{F}$. An invertible linear transformation from V to W is called an isomorphism. If there exists an isomorphism from V to W, we say that V and W are isomorphic.

Exercise 1. *Show that isomorphism is an equivalence relation on finite dimensional vector spaces over the field* $\mathbb{F}$.

Example 7. *Show that* $\mathbb{R}^2(\mathbb{R})$ *and* $\mathbb{C}(\mathbb{R})$ *are isomorphic.*

Solution: Define $T : \mathbb{R}^2 \to \mathbb{C}$ as $T(x,y) = x + iy$. Then T is linear and $Ker(T) = \{(x,y) \in \mathbb{R}^2 \mid x + iy = 0 + 0i\} = \{(0,0)\}$. Hence, T is one-one. Note that $\dim \mathbb{R}^2 = \dim \mathbb{C} = 2$ over $\mathbb{R}$. By rank-nullity theorem, the map is onto.

Definition 8. Let V be a vector space of dimension n. A basis B is called an ordered basis if there is an one to one map between B and the set $\{1,\ldots,n\}$. In simple words, a basis B with an ordering of the elements (of B) is called an ordered basis.

Definition 9. Let V be a vector space with an ordered basis $B = \{v_1, v_2, \ldots, v_n\}$ over the field $\mathbb{F}$. Then for any $v \in V$ there exists a unique $(a_1, a_2, \ldots, a_n) \in \mathbb{F}^n$ such that $v = a_1v_1 + a_2v_2 + \ldots + a_nv_n$. Then the column vector $(a_1, \ldots, a_n)^T$, denoted as $[v]_B$, is called the **coordinate vector** of v with respect to the basis B.

For example, in $\mathbb{F}^n$ the coordinate vector of $(x_1, x_2, \ldots, x_n)$ with respect to the standard basis $\{e_1, \ldots, e_n\}$ is $(x_1, x_2, \ldots, x_n)^T$. Consider $\mathbb{R}^2$ with the basis $B = \{(1,1),(1,-1)\}$. Let $v = (x,y)$. Then $(x,y) = a_1(1,1) + a_2(1,-1)$ if and only if $a_1 = \frac{x+y}{2}$ and $a_2 = \frac{x-y}{2}$. Hence, $[(x,y)]_B = (\frac{x+y}{2}, \frac{x-y}{2})^T = \begin{pmatrix} \frac{x+y}{2} \\ \frac{x-y}{2} \end{pmatrix}$. Consider another basis $B' = \{(1,2),(2,1)\}$. Then $[(x,y)]_{B'} = \begin{pmatrix} \frac{2y-x}{3} \\ \frac{2x-y}{3} \end{pmatrix}$. Thus, the coordinate vector of a vector depends on the basis and it changes with a change of basis.

Theorem 10. Let V be an n-dimensional vector space over $\mathbb{F}$. Then $V \cong \mathbb{F}^n$.

Proof: Let $B = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis of $V(\mathbb{F})$. The map $T : V \to \mathbb{F}^n$ given by $T(v) = [v]_B$ is an isomorphism. First we show that T is linear. Let $v, v' \in V$ with $[v]_B = (a_1, a_2, \ldots, a_n)^T$ and $[v']_B = (b_1, b_2, \ldots, b_n)^T$. Then $\alpha v + \beta v' = (\alpha a_1 + \beta b_1)v_1 + \cdots + (\alpha a_n + \beta b_n)v_n$ so that $[(\alpha v + \beta v')]_B = (\alpha a_1 + \beta b_1, \ldots, \alpha a_n + \beta b_n)^T = \alpha(a_1, \ldots, a_n)^T + \beta(b_1, \ldots, b_n)^T = \alpha T(v) + \beta T(v')$. Now $ker(T) = \{v \mid T(v) = 0\} = \{v \mid [v]_B = 0\} = \{0\}$. Thus T is one-one and onto (rank-nullity theorem).

Corollary 11. Two finite-dimensional vector spaces V and W over the field $\mathbb{F}$ are isomorphic if and only if $\dim(V) = \dim(W)$.