Lecture 10

Linear Transformation

Definition 1. Let V and W be vector spaces over field $\mathbb{F}$. A map $T : V \to W$ is said to be a linear map (or linear transformation) if for $\forall\ \alpha \in \mathbb{F}$ and $\forall\ v_1, v_2 \in V$ we have:

(i) $T(v_1 + v_2) = T(v_1) + T(v_2)$, (ii) $T(\alpha v) = \alpha T(v)$.

Example 2. *1. The map $T : V \to W$ defined by $T(v) = 0$ for all $v \in V$, is linear (the zero map).*

2. The map $T : V \to V$ defined by $T(v) = v$ for all $v \in V$, is linear (the identity map).

3. Let $m \leq n$. Then a map $T : \mathbb{R}^m \to \mathbb{R}^n$, defined by $T(x_1, x_2, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$, $(n-m)$ zeroes, is linear (the inclusion map).

4. Let $m \geq n$. Then a map $T : \mathbb{R}^m \to \mathbb{R}^n$ defined by $T(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_n)$, is linear (the projection map).

5. A map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, -x_2)$, is linear (reflection along x-axis).

6. A map $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_\theta(x, y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$, is linear (rotation about origin with angle θ.

7. Let A be a matrix of order $m \times n$. Then A defines a linear map $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A(x) = Ax$.

8. Let $D : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $D(f(x)) = \frac{d}{dx}f(x)$. Then D is linear (differentiation map).

Proposition 3. Let $T : V \to W$ be a linear map. Then

(i) $T(0) = 0$; (ii) $T(-v) = -T(v)$; (iii) $T(v_1 - v_2) = T(v_1) - T(v_2)$.

Definition 4. Let $T : V \to W$ be a linear map. Then the null space (or kernel) of $T = \{v \in V : T(v) = 0\}$, denoted as $\ker(T)$ and Range space (or Image) of $T = \{T(v) : v \in V\}$ denoted as Range(T).

Example 5. *1. If $T : V \longrightarrow W$ is the zero map, then ker$(T) = V$ and Range$(T) = \{0\}$.*

2. If $T : V \longrightarrow V$ is the identity map, then ker$(T) = \{0\}$ and Range$(T) = V$.

3. If $T : P_n(\mathbb{R}) \longrightarrow P_n(\mathbb{R})$ defined by $T(f(x)) = \frac{d}{dx}(f(x))$, then ker$(T)$ contains all constant polynomials and Range$(T) = P_{n-1}(\mathbb{R})$.

Theorem 6. Let $T : V \to W$ be a linear map. Then *ker*(T) and *Range*(T) are subspaces of V and W respectively. (Prove it yourself!)

Definition 7. The dimension of null space Ker(T) is called the **nullity** of T and the dimension of the range space Range(T) of T is called the **rank** of T.

Theorem 8. *Let V be a finite-dimensional vector space over the field $\mathbb{F}$ and let $\{v_1, \ldots, v_n\}$ be a basis for V. Let W be a vector space over the same field $\mathbb{F}$ and let $w_1, w_2, \ldots, w_n$ be any vectors in W. Then there is precisely one linear transformation T from V to W such that $T(v_i) = w_i \ \forall i = 1, \ldots, n$, and it is given by $T(v) = \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n)$, where $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$.*

Theorem 9. Let $T : V \to W$ be a linear map and $B = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. Then the T is completely determined by its images on basis elements and Range(T) $= L(\{T(v_1), T(v_2), \ldots, T(v_n)\})$.

Proof: Let $v \in V$. Then $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ for some $\alpha_i \in \mathbb{F}$ and $i = 1, \ldots, n$. The map T is linear, $T(v) = T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n)$, that is, image of any vector is a linear combination of images of basis vectors. Thus, Range(T) $= \{T(v) : v \in V\} = \{\alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) : v_1, v_2, \ldots, v_n \in B,\ \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}\} = L(\{T(v_1), T(v_2), \ldots, T(v_n)\})$.

Corollary 10. (Riesz Representation Theorem) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a linear map. Then there exist $a \in \mathbb{R}^n$ such that $T(x) = a^t x$.

Proof: Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Then $T(x) = T(\sum_{i=1}^n x_i e_i) = \sum_{i=1}^n x_i T(e_i)$. Let $T(e_i) = a_i$. Thus $T(x) = a^T x$, where $a = (a_1, \ldots, a_n)$. □

Example 11. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + y - z, x - y + z, y - z)$. The null space of T is $\{(x, y, z) : x + y - z = 0, x - y + z = 0, y - z = 0\}$ which is the solution space of a homogeneous system of linear equations. Thus, $\ker(T) = \{(x, y, z) : x = 0, y = z, z \in \mathbb{R}\} = \{(0, t, t) : t \in \mathbb{R}\} = L(\{(0, 1, 1)\})$. Thus basis of ker(T) is $\{(0, 1, 1)\}$ (as non-zero singleton is independent) so that Nullity(T) = 1. Range(T) $= L(\{T(e_1), T(e_2), T(e_3)\}) = L(\{(1, 1, 0),\ (1, -1, 1),\ (-1, 1, -1)\}) = L(\{(1, 1, 0),\ (1, -1, 1)\}) = \{\alpha(1, 1, 0) + \beta(1, -1, 1) \mid \alpha, \beta \in \mathbb{R}\} = \{(\alpha + \beta, \alpha - \beta, \beta) \mid \alpha, \beta \in \mathbb{R}\}$. Note that Range of T is linear span of $\{(1, 1, 0), (1, -1, 1)\}$ which is linearly independent so that Rank(T) is 2.