Joint Moment generating function

Let $\underline{X} = (X_1, X_2, \cdots, X_n)$ be a $n-$ dimensional random vector and let $A = \{(t_1, t_2, \cdots, t_n) \in \mathbb{R}^n \mid E(e^{\sum_{i=1}^{n} t_i X_i}) \text{ is finite}\}$. The function $M_{\underline{X}} : A \longrightarrow \mathbb{R}$, defined by

$$M_{\underline{X}}(\underline{t}) = E(e^{\sum_{i=1}^{n} t_i X_i}), \ \forall \ \underline{t} = (t_1, t_2, \cdots, t_n) \in A$$

is known as the joint moment generating function (j.m.g.f.) of the random vector $\underline{X}$ if $E(e^{\sum_{i=1}^{n} t_i X_i})$ is finite on a rectangle $(-\underline{a}, \underline{a}) \subseteq A$ for some $(a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$, where $a_i > 0, \ i = 1, 2, \cdots, n$.

Note:

(1) $M_{\underline{X}}(\underline{0}) = 1$, where $\underline{0} = (0, 0, \cdots, 0)$.

(2) If $X_1, X_2, \cdots, X_n$ are independent, then $M_{\underline{X}}(\underline{t}) = E(e^{\sum_{i=1}^{n} t_i X_i}) = E(\prod_{i=1}^{n} e^{t_i X_i}) = \prod_{i=1}^{n} E(e^{t_i X_i})$ $= \prod_{i=1}^{n} M_{X_i}(t_i), \ \forall \ \underline{t} = (t_1, t_2, \cdots, t_n) \in A$, where M_{X_i} is the m.g.f. of $X_i, \ i = 1, 2, \ldots, n$.

Theorem 1. *Let $\underline{X} = (X_1, X_2, \cdots, X_n)$ be a $n-$ dimensional random vector with the joint moment generating function (j.m.g.f.) $M_{\underline{X}}$ that is finite on a rectangle interval $(-\underline{a}, \underline{a}) = (-a_1, a_1) \times (-a_2, a_2) \times \cdots \times (-a_n, a_n) \subseteq \mathbb{R}^n$, where $a_i > 0, \ i = 1, 2, \ldots, n$. Then $M_{\underline{X}}$ possesses partial derivatives of all orders in $(-\underline{a}, \underline{a})$. Furthermore, for positive integers $k_1, k_2, \ldots, k_n$,*

$$E(X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n}) = \left[\frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial t_1^{k_1} \partial t_2^{k_2} \cdots \partial t_n^{k_n}} M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}}, \ \text{where } \underline{t} = (t_1, t_2, \cdots, t_n) \text{ and } \underline{0} = (0, 0, \cdots, 0).$$

In particular,

$$E(X_i) = \left[\frac{\partial}{\partial t_i} M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}}, \ i = 1, 2, \ldots, n;$$
$$E(X_i^m) = \left[\frac{\partial^m}{\partial t_i^m} M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}}, \ i = 1, 2, \ldots, n;$$
$$Var(X_i) = \left[\frac{\partial^2}{\partial t_i^2} M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}} - \left(\left[\frac{\partial}{\partial t_i} M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}}\right)^2, \ i = 1, 2, \ldots, n;$$

and, for $i, j \in \{1, 2, \ldots, n\}, i \neq j$

$$Cov(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j) = \left[\frac{\partial^2}{\partial t_i \partial t_j} M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}} - \left[\frac{\partial}{\partial t_i} M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}} \left[\frac{\partial}{\partial t_j} M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}}.$$

Also

$$M_{\underline{X}}(0, \cdots, 0, t_i, 0, \cdots, 0) = E(e^{t_i X_i}) = M_{X_i}(t_i);$$
$$M_{\underline{X}}(0, \cdots, 0, t_i, 0, \cdots, 0, t_j, 0, \cdots, 0) = E(e^{t_i X_i + t_j X_j}) = M_{X_i, X_j}(t_i, t_j), \ i, j \in \{1, 2, \ldots, n\},$$

provided the involved expectations are finite.

Definition 2. *Let $\underline{X}$ and $\underline{Y}$ be two $n-$ dimensional random vectors with joint c.d.f. $F_{\underline{X}}$ and $F_{\underline{Y}}$ respectively. We say that $\underline{X}$ and $\underline{Y}$ have the same distribution (or are identically distributed) if $F_{\underline{X}}(\underline{x}) = F_{\underline{Y}}(\underline{x}), \ \forall \ \underline{x} \in \mathbb{R}^n$. In this case,it is written as $\underline{X} \stackrel{d}{=} \underline{Y}$.*

Theorem 3. (1) *Let $\underline{X}$ and $\underline{Y}$ be two $n-$ dimensional random vectors with joint p.m.f.'s $f_{\underline{X}}$ and $f_{\underline{Y}}$, respectively. Then, $\underline{X} \stackrel{d}{=} \underline{Y}$ if and only if $f_{\underline{X}}(\underline{x}) = f_{\underline{Y}}(\underline{x})$. $\forall \ \underline{x} \in \mathbb{R}^n$.*

(2) *Let* $\underline{X}$ *and* $\underline{Y}$ *be two* $n-$ *dimensional continuous type random vectors. Then,* $X \overset{\mathrm{d}}{=} Y$ *if and only if there exist versions of joint p.d.f.'s* $f_{\underline{X}}$ *and* $f_{\underline{Y}}$ *of* $\underline{X}$ *and* $\underline{Y}$*, respectively, such that* $f_{\underline{X}}(\underline{x}) = f_{\underline{Y}}(\underline{x})$*.* $\forall\ \underline{x} \in \mathbb{R}^n$*.*

Theorem 4. *Let* $\underline{X}$ *and* $\underline{Y}$ *be two* $n-$ *dimensional random vectors of either discrete type or of continuous type with* $\underline{X} \overset{\mathrm{d}}{=} \underline{Y}$*. Then, for any function* $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ *with* $h^{-1}(A) \in \mathbb{B}_{\mathbb{R}^n}$*, for every* $A \in \mathbb{B}_{\mathbb{R}}$*, we have*

$$h(\underline{X}) \overset{\mathrm{d}}{=} h(\underline{Y})$$

and

$$E(h(\underline{X})) = E(h(\underline{Y})),$$

provided the expectations are finite.

Theorem 5. X_1 *and* X_2 *are independent random variables if and only if* $M_{X_1,X_2}(t_1,t_2) = M_{X_1,X_2}(t_1,0,)M_{X_1,X_2}(0,t_2)$*, for all* $(t_1,t_2) \in \mathbb{R}^2$*.*

Theorem 6. *Let* $\underline{X}$ *and* $\underline{Y}$ *be two* $n-$ *dimensional random vectors of either discrete type or of continuous type with having joint m.g.f.'s* $M_{\underline{X}}$ *and* $M_{\underline{Y}}$*, respectively that are finite on a rectangle* $(-\underline{a},\underline{a}) \subseteq A$ *for some* $(a_1,a_2,\cdots,a_n) \in \mathbb{R}^n$*, where* $a_i > 0,\ i = 1,2,\cdots,n$*. Suppose that* $M_{\underline{X}}(\underline{t}) = M_{\underline{Y}}(\underline{t})$*,* $\forall\ \underline{t} \in (-\underline{a},\underline{a})$*. Then* $\underline{X} \overset{\mathrm{d}}{=} \underline{Y}$*.*

Example 7. *Let* $X_1, X_2, \ldots, X_n$ *be independent random variables such that* $X_i \sim Bin(n_i,\theta),\ 0 < \theta < 1, n_i\{1,2,\cdots\}, i = 1,2,\ldots,n$*. Then show that*

$$\sum_{i=1}^{n} X_i \sim Bin(\sum_{i=1}^{n} n_i, \theta).$$

Solution: Let $Y = \sum\limits_{i=1}^{n} X_i$. Then

$$\begin{aligned} M_Y(t) &= E(e^{t\sum\limits_{i=1}^{n} X_i}) \\ &= E(\prod_{i=1}^{n} e^{tX_i}) \\ &= \prod_{i=1}^{n} E(e^{tX_i}) \\ &= \prod_{i=1}^{n} M_{X_i}(t) \\ &= \prod_{i=1}^{n} (1-\theta+\theta e^t) \\ &= (1-\theta+\theta e^t)^{\sum\limits_{i=1}^{n} n_i},\ \forall\ t \in \mathbb{R} \end{aligned}$$

Since m.g.f. of Bin$(\sum\limits_{i=1}^{n} n_i, \theta)$ is $(1-\theta+\theta e^t)^{\sum\limits_{i=1}^{n} n_i}$, by Theorem 6, $Y \sim$ Bin$(\sum\limits_{i=1}^{n} n_i, \theta)$.