

# Gamma and Exponential Distribution

Consider the improper integral  $\int_0^{\infty} e^{-t} t^{\alpha-1} dt = \int_0^1 e^{-t} t^{\alpha-1} dt + \int_1^{\infty} e^{-t} t^{\alpha-1} dt$ , where  $\alpha \in \mathbb{R}$ . By Limit comparison test,  $\int_0^1 e^{-t} t^{\alpha-1} dt$  converges, for all  $\alpha > 0$  and the  $\int_1^{\infty} e^{-t} t^{\alpha-1} dt$  converges, for all  $\alpha \in \mathbb{R}$ . Hence, the  $\int_0^{\infty} e^{-t} t^{\alpha-1} dt$  is convergent if and only if  $\alpha > 0$ .

**Definition 1.** The function  $\Gamma : (0, \infty) \rightarrow (0, \infty)$ , defined by,

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$$

is called the gamma function.

## Properties:

- (1)  $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ ,  $\alpha > 0$ .
- (2)  $\Gamma(n) = (n - 1)!$ ,  $n \in \mathbb{N}$  with the convention that  $0! = 1$ .
- (3)  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ . In general, for  $n \in \mathbb{N} \cup \{0\}$ , we have  $\Gamma(\frac{2n+1}{2}) = \frac{(2n)!}{n!4^n} \sqrt{\pi}$ .

## 1. GAMMA DISTRIBUTION

A continuous random variable  $X$  is said to have a gamma distribution with parameters  $\alpha > 0$  and  $\lambda > 0$  (written as  $X \sim G(\alpha, \lambda)$ ) if probability density function of  $X$  is given by

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha e^{-\lambda x} x^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

Now, the  $r$ -th moment of  $X \sim G(\alpha, \lambda)$  is

$$\begin{aligned} E(X^r) &= \int_{-\infty}^{\infty} x^r f_X(x) dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^r e^{-\lambda x} x^{\alpha-1} dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} e^{-\lambda x} x^{(\alpha+r)-1} dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)\lambda^{(\alpha+r)}} \int_0^{\infty} e^{-t} t^{(\alpha+r)-1} dt, \text{ (by putting } \lambda x = t) \\ &= \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)\lambda^r} \\ &= \frac{\alpha(\alpha + 1) \cdots (\alpha + r - 1)}{\lambda^r} \end{aligned}$$

Hence

$$\begin{aligned} E(X) &= \frac{\alpha}{\lambda}; \\ E(X^2) &= \frac{\alpha(\alpha + 1)}{\lambda^2}; \\ \text{Var}(X) &= E(X^2) - (E(X))^2 = \frac{\alpha}{\lambda^2}. \end{aligned}$$

The m.g.f. of  $X \sim G(\alpha, \lambda)$  is

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} e^{-(\lambda-t)x} x^{\alpha-1} dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda-t)^\alpha} \int_0^{\infty} e^{-z} z^{\alpha-1} dz, \text{ if } t < \lambda \text{ (by putting } \lambda - t = z) \\ &= \left( \frac{\lambda}{\lambda - t} \right)^\alpha, \text{ if } t < \lambda. \end{aligned}$$

**Remark 2.** Let  $X \sim G(\alpha, \lambda)$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a function defined by  $h(x) = \lambda x$ . Since the support of  $X$  is  $E_X = (0, \infty)$ , the support of  $Z = h(X) = \lambda X$  is  $E_Z = (0, \infty)$ . Clearly,  $h$  is strictly increasing on  $E_X$ . Therefore, the p.d.f. of  $Z = \lambda X$  is

$$\begin{aligned} f_Z(z) &= \begin{cases} f_X(h^{-1}(z)) \left| \frac{d}{dz} h^{-1}(z) \right|, & \text{if } z > 0 \\ 0, & \text{if } z \leq 0 \end{cases} \\ &= \begin{cases} \frac{e^{-z} z^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } z > 0 \\ 0, & \text{if } z \leq 0 \end{cases} \end{aligned}$$

Hence  $Z \sim G(\alpha, 1)$ .

## 2. EXPONENTIAL DISTRIBUTION

A  $G(1, \lambda)$  distribution is called an exponential distribution with parameter  $\lambda > 0$  and it is denoted by  $\text{Exp}(\lambda)$ . Thus p.d.f. of  $\text{Exp}(\lambda)$  is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

If  $X \sim \text{Exp}(\lambda)$ , then

$$\begin{aligned}E(X^r) &= \frac{r!}{\lambda^r}; \\E(X) &= \frac{1}{\lambda}; \\E(X^2) &= \frac{2}{\lambda^2}; \\Var(X) &= E(X^2) - (E(X))^2 = \frac{1}{\lambda^2}; \\M_X(t) &= \frac{\lambda}{\lambda - t}, \text{ if } t < \lambda.\end{aligned}$$

The d.f. of  $X \sim \text{Exp}(\lambda)$  is

$$\begin{aligned}F_X(x) &= \int_{-\infty}^x f_X(t) dt \\&= \begin{cases} \int_0^x \lambda e^{-\lambda t}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases} \\&= \begin{cases} 1 - e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}\end{aligned}$$

- Remark 3.** (1) *A Poisson Process is a model for a series of discrete event where the average time between events is known, but the exact timing of events is random.*  
(2) *The exponential distribution occurs naturally if we consider the distribution of the length of intervals between successive events in a Poisson process or, equivalently, the distribution of the interval (i.e. the waiting time) before the first event.*

**Example 4.** *The waiting time for occurrence of an event  $E$  (say repair time of a machine) is exponentially distributed with mean of 30 minutes. Find the conditional probability that the waiting time for occurrence of event  $E$  is at least 5 hours given that it has not occurred in the first 3 hours.*

**Solution:** Let  $X$  be the waiting time (in hours) for the occurrence of event  $E$ . Then  $X \sim \text{Exp}(2)$ . Hence, the required probability is  $P(\{X > 5\}|\{X > 3\}) = \frac{P(\{X > 5\})}{P(\{X > 3\})} = \frac{e^{-10}}{e^{-6}} = e^{-4}$ .