GENERALIZED BI-CIRCULAR IDEMPOTENTS ON SOME SPACES OF ANALYTIC FUNCTIONS

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Abstract. Let X be a complex normed space. A map $P : X \to X$ is called idempotent if $P^2 = P$. A collection $\mathcal{C} = \{P_1, P_2\}$ of nonzero distinct orthogonal ($P_1P_2 = P_2P_1 = 0$), not necessarily linear, idempotent maps on X is said to be a family of generalized bi-circular idempotents if there exist distinct unit modulus complex numbers λ_1, λ_2 such that $P_1 + P_2 = I$ (identity operator on X) and $\lambda_1 P_1 + \lambda_2 P_2$ is a (not necessarily linear) surjective isometry on X. This generalizes the notion of generalized bi-circular projections on Banach spaces introduced by Fošner, Ilišević and Li [5] to nonlinear maps. In this paper, we give complete descriptions of generalized bi-circular idempotents on the following complex normed linear spaces: the space $\mathcal{S}_A$ of analytic functions f on the open unit disc $\mathbb{D}$ whose derivative can be extended to the closed unit disc $\overline{\mathbb{D}}$, and the space $\mathcal{S}^\infty$ of analytic functions f on $\mathbb{D}$ with bounded derivatives.

1. Introduction and Basic Results

Let X be a complex normed space. A map $P : X \to X$ is called idempotent if $P^2 = P$. A bounded (continuous) and idempotent linear operator on X is called a projection. In this note, we extend a class of bi-contractive projections on Banach spaces, known as generalized bi-circular projections and introduced by Fošner, Ilišević and Li [5], to a nonlinear setting and study some of its properties. A projection P is said to be a generalized bi-circular projection if $T = P + \lambda(I - P)$ is a surjective isometry on X for some $\lambda \in \mathbb{T} \setminus \{1\}$, where I denotes the identity operator on X and $\mathbb{T}$ denotes the unit circle in the complex numbers $\mathbb{C}$.

Over the last two decades, these classes of projections have been extensively studied for several Banach spaces. We refer interested readers to the papers [1, 2, 5, 6, 7] and the references therein.

To extend this notion to nonlinear maps, i.e., both P (which is simply an idempotent map) and the isometry T, are not necessarily linear, we first observe that if P is an idempotent map on X, then $I - P$ may not be idempotent. A simple example is to define $P : X \to X$ as $P(x) = \frac{x}{\|x\|}$ when $x \neq 0$ and $P(0) = 0$. We will revisit this example later. Therefore, in our definition, we must assume that $I - P$ is also idempotent. Thus, we have the following definition.

Definition 1.1. *Let X be a complex normed space. A collection $\mathcal{C} = \{P_1, P_2\}$ of nonzero distinct idempotent maps on X is said to be a family of* ***generalized bi-circular idempotents*** *(GBI, for short) if*

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(1) *$P_1 + P_2 = I$,*

(2) *$P_1P_2 = 0 = P_2P_1$, and*

(3) *there exist distinct λ_1, $\lambda_2 \in \mathbb{T}$ such that $T = \lambda_1 P_1 + \lambda_2 P_2$ is a surjective isometry on X.*

Each P_i, $i = 1, 2$, is called a generalized bi-circular idempotent. We also say that $\mathcal{C} = \{P_1, P_2\}$ is a family of generalized bi-circular idempotents corresponding to the isometry T. We sometimes refer to T as the isometry associated with the family $\mathcal{C}$.

We recall that a map $T : X \to X$ is called an isometry if $\|Tx - Ty\| = \|x - y\|$ for all $x, y \in X$. Moreover, if T is an isometry, then T^n is also an isometry, for $n \geq 2$. We consider the Hardy space $H^\infty(\mathbb{D})$, which is the commutative Banach algebra of all bounded analytic functions on $\mathbb{D}$, and the disc algebra $A(\overline{\mathbb{D}})$, which is the commutative Banach algebra of all analytic functions on $\mathbb{D}$ that can be continuously extended to $\overline{\mathbb{D}}$; both equipped with the supremum norm

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

We define $\mathcal{S}_A$ to be the Banach space of all analytic functions f on $\mathbb{D}$ whose derivative f' belongs to $A(\overline{\mathbb{D}})$, and $\mathcal{S}^\infty$ to be the Banach space of all analytic functions f on $\mathbb{D}$ whose derivative f' belongs to $H^\infty(\mathbb{D})$; both endowed with the norm

$$\|f\| = |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)|.$$

In this paper, we give complete descriptions of GBIs on the spaces $\mathcal{S}_A$ and $\mathcal{S}^\infty$. We also find out the structure of isometric reflections on these spaces. An isometric reflection on a normed space X is an isometry T such that $T^2 = I$.

It is important to note that in [3], Botelho and Miura initially introduced the notion of generalized bi-circular idempotents as follows. An idempotent map P is called a generalized bi-circular idempotent if there exists a unimodular complex number $\lambda \neq 1$ such that $P + \lambda(I - P)$ is a surjective isometry. Moreover, in [3, 4], the authors studied generalized bi-circular idempotents on the space of continuously differentiable complex-valued functions on $[0, 1]$.

In our definition of generalized bi-circular idempotents (Definition 1.1), we have emphasized that $I - P$ is also an idempotent map. Unlike projections (P is a projection if and only if $I - P$ is a projection), this is a crucial assumption for the case of idempotent maps. Our next definition and lemma address this issue.

Definition 1.2. *A map P on a normed space X is said to be **bi-potent** if both P and $I - P$ are idempotent maps.*

Lemma 1.3. *Let P be a map on a normed space X. If P is idempotent, then $(I - P)P = 0$. Further, P is bi-potent if and only if $P(I - P) = 0$.*

Proof. Let P be an idempotent map. Then $(I - P)P = P - P^2 = P - P = 0$.

Suppose that P is bi-potent. Then for any $x \in X$, $(I - P)(I - P)x = (I - P)x$. This implies that $I(I - P)x - P(I - P)x = (I - P)x$. It follows that $P(I - P)x = 0$. Hence, $P(I - P) = 0$.

Conversely, let $x \in X$. Then $(I-P)^2x = (I-P)(I-P)x = I(I-P)x - P(I-P)x = (I-P)x$. Thus, $I-P$ is idempotent. $\square$

An example of an idempotent map that is not bi-potent is given below.

Example 1.4. *Let $P : X \to X$ be defined as $P(x) = \frac{x}{\|x\|}$ when $x \neq 0$ and $P(0) = 0$. Then P is idempotent, but $I-P$ is not. To verify the later claim, consider a nonzero $x \in X$ such that $\|x\| \neq 1$. Then $(I-P)x = x - \frac{x}{\|x\|} \neq 0$. It follows that $P(I-P)x \neq 0$. Therefore, $I-P$ is not an idempotent map.*

Lemma 1.5. *Let $\mathcal{C} = \{P_1, P_2\}$ be a family of generalized bi-circular idempotents such that $T = \lambda_1 P_1 + \lambda_2 P_2$. Then $TP_1 = \lambda_1 P_1$ and $TP_2 = \lambda_2 P_2$.*

Proof. Multiplying both sides by P_1 and P_2 and using the orthogonality of P_1 and P_2, we obtain the desired result. $\square$

Consider a family $\mathcal{C} = \{P_1, P_2\}$ of GBI corresponding to the isometry T. The Mazur-Ulam theorem [8] states that some translation of T is a real linear isometry. In other words, $T = T(0) + S$, where S is a real linear isometry on X. Our next proposition establishes that T is a real linear.

Proposition 1.6. *Let $\mathcal{C} = \{P_1, P_2\}$ be a family of generalized bi-circular idempotents corresponding to the isometry T. Then $T(0) = 0$. Moreover, T is a real linear.*

Proof. Let $T = \lambda_1 P_1 + \lambda_2 P_2$, where λ_1, λ_2 are distinct complex numbers of modulus one. By the Mazur-Ulam theorem, we can write $T = T(0) + S$, where S is a real linear surjective isometry. By Lemma 1.5, we have $TP_1 = \lambda_1 P_1$ and $TP_2 = \lambda_2 P_2$. Consequently, $T(0) + SP_1 = \lambda_1 P_1$ and $T(0) + SP_2 = \lambda_2 P_2$. Adding these two equations, we get $2T(0) + SP_1 + SP_2 = \lambda_1 P_1 + \lambda_2 P_2$. Since S is real linear and $P_1 + P_2 = I$, it follows that $2T(0) + S = T(0) + S$. Hence, $T(0) = 0$, and thus, T is real linear. $\square$

Corollary 1.7. *Let $\mathcal{C} = \{P_1, P_2\}$ be a family of generalized bi-circular idempotents on a normed space X. Then P_1 and P_2 are real linear.*

Proof. Let $T = \lambda_1 P_1 + \lambda_2 P_2$, for some distinct $\lambda_1, \lambda_2 \in \mathbb{T}$. Since $P_1 + P_2 = I$, we conclude that

$$P_1 = \frac{T - \lambda_2 I}{\lambda_1 - \lambda_2} \text{ and } P_2 = \frac{T - \lambda_1 I}{\lambda_2 - \lambda_1}. \tag{1.1}$$

Since T is real linear, P_1 and P_2 automatically become real linear. $\square$

Remark 1.8. *The above Proposition and Corollary only imply that the maps under consideration are real linear, and not complex linear. So, essentially, they are nonlinear maps.*

Example 1.9. *Define the real linear idempotents $P_1, P_2 : \mathbb{C} \to \mathbb{C}$, by*

$$P_1(z = x + iy) = x, \quad \textit{and} \quad P_2(z = x + iy) = iy \quad \textit{for all} \quad x, y \in \mathbb{R}.$$

Obviously, $P_1 + P_2 = I$ and $P_1P_2 = 0 = P_2P_1$. For any $x, y \in \mathbb{R}$, let

$$T(x+iy) = \lambda_1 P_1 + \lambda_2 P_2 = \lambda_1 x + i\lambda_2 y,$$

where $\lambda_1 \neq \lambda_2 \in \mathbb{T}$. It follows that

$$|T(z)|^2 = |\lambda_1|^2 x^2 + |\lambda_2|^2 y^2 + xy\big(-i\lambda_1\overline{\lambda_2} + i\overline{\lambda_1}\lambda_2\big) = x^2 + y^2 - 2xy\,\Im\big(\overline{\lambda_1}\lambda_2\big),$$

where $\Im\big(\overline{\lambda_1}\lambda_2\big)$ denotes the imaginary part of $\overline{\lambda_1}\lambda_2$. Now, T is an isometry if and only if $|T(z)|^2 = |z|^2 = x^2 + y^2$, for all $x, y \in \mathbb{R}$. Hence, $\Im\big(\overline{\lambda_1}\lambda_2\big) = 0$. Let $\overline{\lambda_1}\lambda_2 = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Thus,

$$\sin\theta = 0 \quad \Longrightarrow \quad \theta = n\pi, n \in \mathbb{Z} \quad \Longrightarrow \quad \overline{\lambda_1}\lambda_2 = \cos\theta = (-1)^n = \pm 1.$$

This concludes that $\lambda_1 = -\lambda_2$. So, $T(x+iy) = -\lambda_2 x + i\lambda_2 y = -\lambda_2(x - iy) = -\lambda_2\overline{z}$. Clearly, T is a surjective isometry on $\mathbb{C}$. Therefore, the collection $\{P_1, P_2\}$ is a family of generalized bi-circular idempotents for $\lambda_1 = -\lambda_2$.

The structure of surjective isometries on the spaces $\mathcal{S}_A$ and $\mathcal{S}^\infty$ was characterized by Miura and Niwa in [9] and [10] respectively, and is stated in the following theorem.

Theorem 1.10. [9, Theorem 1] [10, Theorem 2] *Let $\mathcal{X} = \mathcal{S}_A$ or $\mathcal{S}^\infty$. If $T : \mathcal{X} \to \mathcal{X}$ is a surjective, not necessarily linear, isometry with respect to the norm $\|f\| = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|$ for $f \in \mathcal{X}$, then there exist constants $c_0, c_1, \mu \in \mathbb{T}$ and $a \in \mathbb{D}$ such that for all $f \in \mathcal{X}$, and $z \in \mathbb{D}$*

$$Tf(z) = T(0)(z) + c_0 f(0) + \int_{[0,z]} c_1 f'(\psi(\xi))d\xi, \ \textit{or} \qquad \text{(Form I)}$$

$$Tf(z) = T(0)(z) + \overline{c_0 f(0)} + \int_{[0,z]} c_1 f'(\psi(\xi))d\xi, \ \textit{or} \qquad \text{(Form II)}$$

$$Tf(z) = T(0)(z) + c_0 f(0) + \int_{[0,z]} \overline{c_1 f'(\psi(\overline{\xi}))}d\xi, \ \textit{or} \qquad \text{(Form III)}$$

$$Tf(z) = T(0)(z) + \overline{c_0 f(0)} + \int_{[0,z]} \overline{c_1 f'(\psi(\overline{\xi}))}d\xi, \qquad \text{(Form IV)}$$

where $\psi(z) = \mu\frac{z-a}{\bar{a}z-1}$ for all $z \in \mathbb{D}$.

2. Isometric Reflections on $\mathcal{X}$

A map T on a normed linear space X is said to be of the order n if $T^n = I$ for some positive integer n. If $n = 2$, T is called a reflection. If T is an isometry on X of order 2, it is termed as an isometric reflection.

In this section, we find the structure of the isometric reflections on $\mathcal{X} = \mathcal{S}_A$ and $\mathcal{S}^\infty$. We will see in the next section that the isometry associated with a family of generalized bi-circular idempotents is actually an isometric reflection.

Theorem 2.1. *Let T be an isometric reflection on $\mathcal{X}$ of the form (I). Then $c_0, c_1 \in \{\pm 1\}$ and $\psi^2 = I$.*

Proof. An isometry on $\mathcal{X}$ of form (I) is given by

$$Tf(z) = T(0)(z) + c_0 f(0) + \int_{[0,z]} c_1 f'(\psi(\xi))d\xi, \ \forall \ f \in \mathcal{X} \text{ and } z \in \mathbb{D}.$$

This implies that

$$Tf(0) = T(0)(0) + c_0 f(0) \text{ and } (Tf)'(z) = (T(0))'(z) + c_1 f'(\psi(z)).$$

Since T is an isometric reflection, we have $T^2 f(z) = f(z)$ for all $f \in \mathcal{X}$ and $z \in \mathbb{D}$. It follows that

$$T(0)(z) + c_0(Tf)(0) + \int_{[0,z]} c_1 (Tf)'(\psi(\xi))d\xi = f(z).$$

Substituting the values of $T(f)(0)$ and $(Tf)'(\psi(\xi))$ in the above equation, we obtain

$$T(0)(z) + c_0 \left[T(0)(0) + c_0 f(0)\right] + c_1 \int_{[0,z]} \left[(T(0))'(\psi(\xi)) + c_1 f'(\psi^2(\xi))\right] d\xi = f(z). \tag{2.1}$$

Choosing $f = 0$, we get

$$T(0)(z) + c_0 T(0)(0) + c_1 \int_{[0,z]} (T(0))'(\psi(\xi))d\xi = 0.$$

Thus, Equation (2.1) becomes

$$c_0^2 f(0) + c_1^2 \int_{[0,z]} f'(\psi^2(\xi))d\xi = f(z). \tag{2.2}$$

Now, by choosing $f = 1$ we conclude that $c_0^2 = 1$. Thus, $c_0 = \pm 1$.

Differentiating equation (2.2), we get $c_1^2 f'(\psi^2(z)) = f'(z)$. This implies that $c_1 = \pm 1$ and $\psi^2(z) = z$. Hence, the proof is complete. □

Theorem 2.2. *Let T be an isometric reflection on $\mathcal{X}$ of the form (II). Then $c_1 \in \{\pm 1\}$ and $\psi^2 = I$.*

Proof. An isometry of form (II) is given by

$$Tf(z) = T(0)(z) + \overline{c_0 f(0)} + \int_{[0,z]} c_1 f'(\psi(\xi))d\xi, \ \forall \ f \in \mathcal{X} \text{ and } z \in \mathbb{D}.$$

Thus,

$$Tf(0) = T(0)(0) + \overline{c_0 f(0)} \text{ and } (Tf)'(z) = (T(0))'(z) + c_1 f'(\psi(z)).$$

Since T is an isometric reflection, $T^2 f(z) = f(z)$ for all $f \in \mathcal{X}$ and $z \in \mathbb{D}$. This implies that

$$T(0)(z) + \overline{c_0 (Tf)(0)} + \int_{[0,z]} c_1 (Tf)'(\psi(\xi))d\xi = f(z),$$

or

$$T(0)(z) + \overline{c_0 T(0)(0)} + f(0) + c_1 \int_{[0,z]} \left[(T(0))'(\psi(\xi)) + c_1 f'(\psi^2(\xi))\right] d\xi = f(z). \tag{2.3}$$

Choosing $f = 0$, we get

$$T(0)(z) + \overline{c_0 T(0)(0)} + c_1 \int_{[0,z]} (T(0))'(\psi(\xi)) d\xi = 0.$$

So, Equation (2.3) takes the form

$$f(0) + c_1^2 \int_{[0,z]} f'(\psi^2(\xi)) d\xi = f(z).$$

After differentiating this equation, we get $c_1^2 f'(\psi^2(z)) = f'(z)$. It follows that $c_1 = \pm 1$ and $\psi^2(z) = z$. This completes the proof. □

Theorem 2.3. *Let T be an isometric reflection on $\mathcal{X}$ of the form (III). Then $c_0 \in \{\pm 1\}$, $\mu a = \overline{a}$ and $\psi(\overline{\psi(\overline{z})}) = z$ for all $z \in \mathbb{D}$.*

Proof. The form of T is given by

$$Tf(z) = T(0)(z) + c_0 f(0) + \int_{[0,z]} \overline{c_1 f'(\psi(\overline{\xi}))} d\xi \quad \forall\ f \in \mathcal{X} \text{ and } z \in \mathbb{D}.$$

It follows that

$$Tf(0) = T(0)(0) + c_0 f(0) \text{ and } (Tf)'(z) = (T(0))'(z) + \overline{c_1 f'(\psi(\overline{z}))}.$$

Since $T^2 f(z) = f(z)$ for all $f \in \mathcal{X}$ and $z \in \mathbb{D}$, we have

$$T(0)(z) + c_0 (Tf)(0) + \int_{[0,z]} \overline{c_1 (Tf)'(\psi(\overline{\xi}))} d\xi = f(z).$$

This implies that

$$T(0)(z) + c_0 T(0)(0) + c_0^2 f(0) + \int_{[0,z]} \left[\overline{c_1 (T(0))'(\psi(\overline{\xi}))} + f'(\psi(\overline{\psi(\overline{\xi})}))\right] d\xi = f(z). \tag{2.4}$$

Selecting $f = 0$,

$$T(0)(z) + c_0 T(0)(0) + \int_{[0,z]} \overline{c_1 (T(0))'(\psi(\overline{\xi}))} d\xi = 0.$$

So, Equation (2.4) becomes

$$c_0^2 f(0) + \int_{[0,z]} f'(\psi(\overline{\psi(\overline{\xi})})) d\xi = f(z).$$

Hence, $c_0 = \pm 1$ and $\psi(\overline{\psi(\overline{z})}) = z$. Since $\psi(z) = \mu \frac{z-a}{\bar{a}z-1}$, putting $z = \overline{a}$ in the identity $\psi(\overline{\psi(\overline{z})}) = z$, we obtain $\psi(0) = \overline{a}$. This implies that $\mu a = \overline{a}$. Thus, the proof is complete. □

Theorem 2.4. *Let T be an isometric reflection on $\mathcal{X}$ of the form (IV). Then $\mu a = \overline{a}$ and $\psi(\overline{\psi(\overline{z})}) = z$ for all $z \in \mathbb{D}$.*

Proof. The isometry T is given by

$$Tf(z) = T(0)(z) + \overline{c_0 f(0)} + \int_{[0,z]} \overline{c_1 f'(\psi(\overline{\xi}))} d\xi, \ \forall \ f \in \mathcal{X} \text{ and } z \in \mathbb{D}.$$

It is immediate that

$$Tf(0) = T(0)(0) + \overline{c_0 f(0)} \text{ and } (Tf)'(z) = (T(0))'(z) + \overline{c_1 f'(\psi(\overline{z}))}.$$

Moreover, $T^2 f(z) = f(z)$ implies that

$$T(0)(z) + \overline{c_0 (Tf)(0)} + \int_{[0,z]} \overline{c_1 (Tf)'(\psi(\overline{\xi}))} d\xi = f(z).$$

Therefore,

$$T(0)(z) + \overline{c_0 T(0)(0)} + f(0) + \int_{[0,z]} \left[\overline{c_1 (T(0))'(\psi(\overline{\xi}))} + f'(\psi(\overline{\psi(\overline{\xi})})) \right] d\xi = f(z). \tag{2.5}$$

Taking $f = 0$, we obtain

$$T(0)(z) + \overline{c_0 T(0)(0)} + \int_{[0,z]} \overline{c_1 (T(0))'(\psi(\overline{\xi}))} d\xi = 0.$$

Thus, Equation (2.5) takes the form

$$f(0) + \int_{[0,z]} f'(\psi(\overline{\psi(\overline{\xi})})) d\xi = f(z).$$

Therefore, $\psi(\overline{\psi(\overline{z})}) = z$. This implies that $\mu a = \overline{a}$, which completes the proof. □

3. Structure of Generalized bi-circular idempotents on $\mathcal{X}$

In this section, we characterize generalized bi-circular idempotents on $\mathcal{X} = \mathcal{S}_A$ and $\mathcal{S}^\infty$.

Theorem 3.1. *If the collection $\mathcal{C} = \{P_1, P_2\}$ is a family of generalized bi-circular idempotents on $\mathcal{X}$ corresponding to an isometry T of the form (I), then one of the following holds:*

1. *$\mathcal{C}$ is a family of generalized bi-circular projections such that each P_i is the average of the identity operator and an isometric reflection. Moreover, we have $\lambda_1 + \lambda_2 = 0$, $c_0 = \pm\lambda_1$, $c_1 = \pm\lambda_1$ and $\psi^2(z) = z$ for all $z \in \mathbb{D}$.*
2. *$\mathcal{C}$ is a family of bi-circular projections.*

Proof. Let $\mathcal{C} = \{P_1, P_2\}$ be a family of generalized bi-circular idempotents on $\mathcal{X}$ corresponding to an isometry T of form (I). Then $T = \lambda_1 P_1 + \lambda_2 P_2$, where $\lambda_1, \lambda_2 \in \mathbb{T}$. Proposition 1.6 implies that $T(0) = 0$. Thus, T takes the form

$$Tf(z) = c_0 f(0) + \int_{[0,z]} c_1 f'(\psi(\xi)) d\xi, \ \forall \ f \in \mathcal{X} \text{ and } z \in \mathbb{D}.$$

We observe that T is a linear surjective isometry on $\mathcal{X}$. This implies that P_1, P_2 are projections. If follows that $(T - \lambda_1 I)(T - \lambda_2 I) = 0$ or

$$T^2f(z) - (\lambda_1 + \lambda_2)Tf(z) + \lambda_1\lambda_2 f(z) = 0,\ \forall\ f \in \mathcal{X} \text{ and } z \in \mathbb{D}. \tag{3.1}$$

Further,

$$P_i = \frac{T - \lambda_j I}{\lambda_i - \lambda_j},\ i, j = 1, 2,\ i \neq j. \tag{3.2}$$

Now, using the form of T in Equation (3.1), we obtain

$$\left[c_0^2 f(0) + c_1^2 \int_{[0,z]} f'(\psi^2(\xi))d\xi\right] - (\lambda_1 + \lambda_2)\left[c_0 f(0) + c_1 \int_{[0,z]} f'(\psi(\xi))d\xi\right] + \lambda_1\lambda_2 f(z) = 0. \tag{3.3}$$

Choosing $f = 1$ in the above equation, we get

$$c_0^2 - (\lambda_1 + \lambda_2)c_0 + \lambda_1\lambda_2 = 0 \implies c_0 = \lambda_1, \lambda_2.$$

Again, choosing $f = id$, we have

$$c_1^2 z - (\lambda_1 + \lambda_2)c_1 z + \lambda_1\lambda_2 z = 0 \implies c_1 = \lambda_1, \lambda_2.$$

Differentiating Equation (3.3),

$$c_1^2 f'(\psi^2(z)) - (\lambda_1 + \lambda_2)c_1 f'(\psi(z)) + \lambda_1\lambda_2 f'(z) = 0.$$

We claim that $\psi^2(z) = z$ for all $z \in \mathbb{D}$. To see this, we observe that if $\psi(z) = z$ for all $z \in \mathbb{D}$, then the claim is obvious. Suppose that $\psi(z) \neq z$ for some $z \in \mathbb{D}$. Then $z, \psi(z)$ and $\psi^2(z)$ cannot be all distinct. If they are, we choose a function $f \in \mathcal{X}$ such that $f'(z) = 1$ and $f'(\psi(z)) = f'(\psi^2(z)) = 0$. This implies that $\lambda_1\lambda_2 = 0$, a contradiction. Therefore, $\psi^2(z) = z$ for all $z \in \mathbb{D}$.

Now, suppose $\psi(z) \neq z$ for some $z \in \mathbb{D}$. We choose a function $f \in \mathcal{X}$ such that $f'(z) = 0$ and $f'(\psi(z)) = 1$ to obtain $c_1(\lambda_1 + \lambda_2) = 0$. Since $c_1 \in \mathbb{T}$, we conclude $\lambda_1 + \lambda_2 = 0$. Hence, $c_0 = \pm\lambda_1$ and $c_1 = \pm\lambda_1$.

So, Equation (3.2) becomes

$$P_i = \frac{T - \lambda_j I}{\lambda_i - \lambda_j} = \frac{T + \lambda_i I}{2\lambda_i} = \frac{I + S}{2},$$

where $S = \frac{1}{\lambda_i}T$. It is clear that $S^2 = I$.

This proves the first assertion.

If $\psi(z) = z$ for all $z \in \mathbb{D}$, then $Tf(z) = c_0 f(0) + c_1(f(z) - f(0))$. Therefore,

$$P_i f(z) = \frac{(c_0 - c_1)f(0) + (c_1 - \lambda_j)f(z)}{\lambda_i - \lambda_j}.$$

Consequently, for all $\alpha, \beta \in \mathbb{T}$, $f \in \mathcal{X}$ and $z \in \mathbb{D}$, we have

$$\begin{aligned}\alpha P_1 f(z) + \beta P_2 f(z) &= \alpha \frac{(c_0 - c_1) f(0) + (c_1 - \beta) f(z)}{\alpha - \beta} + \beta \frac{(c_0 - c_1) f(0) + (c_1 - \alpha) f(z)}{\beta - \alpha} \\ &= c_0 f(0) + c_1 (f(z) - f(0)),\end{aligned}$$

which is clearly a surjective isometry on $\mathcal{X}$. It follows that the collection $\mathcal{C}$ forms a family of bi-circular projections. Thus, the proof is complete. □

Theorem 3.2. *Suppose the collection $\mathcal{C} = \{P_1, P_2\}$ is a family of generalized bi-circular idempotents on $\mathcal{X}$ corresponding to an isometry T of the form (II). Then one of the following holds:*

(1) *Each P_i is the average of the identity operator and an isometric reflection. Moreover, we have $\lambda_1 + \lambda_2 = 0$, $c_1 = \pm\lambda_1$ and $\psi^2(z) = z$ for all $z \in \mathbb{D}$.*
(2) *$\mathcal{C}$ is a family of bi-circular idempotents.*

Proof. Let $\mathcal{C} = \{P_1, P_2\}$ be a family of generalized bi-circular idempotents on $\mathcal{X}$ corresponding to an isometry T of form (II). Then $T = \lambda_1 P_1 + \lambda_2 P_2$, where $\lambda_1, \lambda_2 \in \mathbb{T}$. Proposition 1.6 implies that $T(0) = 0$. Thus, T takes the form

$$Tf(z) = \overline{c_0 f(0)} + \int_{[0,z]} c_1 f'(\psi(\xi)) d\xi, \ \forall\ f \in \mathcal{X} \text{ and } z \in \mathbb{D}.$$

Further,

$$P_i f(z) = \frac{Tf(z) - \lambda_j f(z)}{\lambda_i - \lambda_j} = \frac{1}{\lambda_i - \lambda_j} \left[\overline{c_0 f(0)} + \int_{[0,z]} c_1 f'(\psi(\xi)) d\xi - \lambda_j f(z) \right], \tag{3.4}$$

$i, j = 1, 2,\ i \neq j$.

On differentiating for $i = 1$, we get

$$(P_1 f)'(z) = \frac{1}{\lambda_1 - \lambda_2} \left[c_1 f'(\psi(z)) - \lambda_2 f'(z) \right].$$

Taking $z = 0$ in the equation (3.4), we have

$$P_1 f(0) = \frac{1}{\lambda_1 - \lambda_2} \left[\overline{c_0 f(0)} - \lambda_2 f(0) \right].$$

Since P_i, $i = 1, 2$, is a generalized bi-circular idempotent, we also have the identity

$$TP_i f(z) = \lambda_i P_i f(z),\ i = 1, 2,\ \forall\ f \in \mathcal{X} \text{ and } z \in \mathbb{D}.$$

It follows that

$$\begin{aligned}&\overline{c_0 (P_1 f)(0)} + \int_{[0,z]} c_1 (P_1 f)'(\psi(\xi)) d\xi \\ &\qquad = \frac{\lambda_1}{\lambda_1 - \lambda_2} \left[\overline{c_0 f(0)} + \int_{[0,z]} c_1 f'(\psi(\xi)) d\xi - \lambda_2 f(z) \right].\end{aligned}$$

Putting the values of $P_1 f(0)$ and $(P_1 f)'(z)$, we have

$$\frac{\overline{c_0}}{\overline{\lambda_1}-\overline{\lambda_2}}\left[\overline{\overline{c_0 f(0)}-\lambda_2 f(0)}\right]+\frac{c_1}{\lambda_1-\lambda_2}\int_{[0,z]}\left[c_1 f'(\psi^2(\xi))-\lambda_2 f'(\psi(\xi))\right]d\xi$$
$$=\frac{\lambda_1}{\lambda_1-\lambda_2}\left[\overline{c_0 f(0)}+\int_{[0,z]}c_1 f'(\psi(\xi))d\xi-\lambda_2 f(z)\right].$$

or

$$\frac{\overline{c_0}}{\overline{\lambda_1}-\overline{\lambda_2}}\left[c_0 f(0)-\overline{\lambda_2 f(0)}\right]+\frac{c_1}{\lambda_1-\lambda_2}\int_{[0,z]}\left[c_1 f'(\psi^2(\xi))-\lambda_2 f'(\psi(\xi))\right]d\xi$$
$$=\frac{\lambda_1}{\lambda_1-\lambda_2}\left[\overline{c_0 f(0)}+\int_{[0,z]}c_1 f'(\psi(\xi))d\xi-\lambda_2 f(z)\right].$$

After simplification, we get

$$\frac{1}{\overline{\lambda_1}-\overline{\lambda_2}}f(0)+\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}f(z)+\frac{c_1}{\lambda_1-\lambda_2}\int_{[0,z]}\left[c_1 f'(\psi^2(\xi))-(\lambda_1+\lambda_2)f'(\psi(\xi))\right]d\xi=0.$$

Differentiating the above equation

$$c_1^2 f'(\psi^2(z))-c_1(\lambda_1+\lambda_2)f'(\psi(z))+\lambda_1\lambda_2 f'(z)=0.$$

Choosing $f'=1$ in the above equation

$$c_1^2-c_1(\lambda_1+\lambda_2)+\lambda_1\lambda_2=0 \implies c_1=\lambda_1,\lambda_2$$

Proceeding in the same way as we did in Theorem 3.1, we conclude that $\psi^2(z)=z$ for all $z\in\mathbb{D}$.

If $\psi(z)\neq z$ for some $z\in\mathbb{D}$, then $\lambda_1+\lambda_2=0$. Moreover, $c_1=\pm\lambda_1$.

Equation (3.4) implies that

$$P_i=\frac{T-\lambda_j I}{\lambda_i-\lambda_j}=\frac{T+\lambda_i I}{2\lambda_i}=\frac{I+S}{2},$$

where $S=\frac{1}{\lambda_i}T$. Clearly, $S^2=I$.

This proves the first assertion.

If $\psi(z)=z$ for all $z\in\mathbb{D}$, then $Tf(z)=\overline{c_0 f(0)}+c_1(f(z)-f(0)$. Thus,

$$P_i f(z)=\frac{\overline{c_0 f(0)}-c_1 f(0)+(c_1-\lambda_j)f(z)}{\lambda_i-\lambda_j}.$$

Now, for all $\alpha, \beta \in \mathbb{T}$, $f \in \mathcal{X}$ and $z \in \mathbb{D}$, we have

$$\begin{aligned}\alpha P_1 f(z) + \beta P_2 f(z) &= \alpha \frac{\overline{c_0 f(0)} - c_1 f(0) + (c_1 - \beta) f(z)}{\alpha - \beta} + \\ &\quad \beta \frac{\overline{c_0 f(0)} - c_1 f(0) + (c_1 - \alpha) f(z)}{\beta - \alpha} \\ &= \overline{c_0 f(0)} + c_1 (f(z) - f(0).\end{aligned}$$

This is indeed a surjective isometry on $\mathcal{X}$. Therefore, the collection $\mathcal{C}$ constitutes a family of bi-circular of idempotents. This completes the proof. $\square$

Theorem 3.3. *Let $\mathcal{C} = \{P_1, P_2\}$ be a family of generalized bi-circular idempotents on $\mathcal{X}$ corresponding to an isometry T of form (III). Then $\mathcal{C}$ is a family of bi-circular idempotents. Moreover, $c_0 = \lambda_1, \lambda_2$ and $\psi(\overline{\psi(\overline{z})}) = z$ for all $z \in \mathbb{D}$.*

Proof. Let $\mathcal{C} = \{P_1, P_2\}$ be a family of generalized bi-circular idempotents on $\mathcal{X}$ corresponding to an isometry T of form (III). Then $T = \lambda_1 P_1 + \lambda_2 P_2$, where $\lambda_1, \lambda_2 \in \mathbb{T}$. Proposition 1.6 implies that $T(0) = 0$. Thus, T takes the form

$$Tf(z) = c_0 f(0) + \int_{[0,z]} \overline{c_1 f'(\psi(\overline{\xi}))} d\xi, \ \forall \ f \in \mathcal{X} \text{ and } z \in \mathbb{D}.$$

We also have,

$$P_i f(z) = \frac{Tf(z) - \lambda_j f(z)}{\lambda_i - \lambda_j} = \frac{1}{\lambda_i - \lambda_j} \left[c_0 f(0) + \int_{[0,z]} \overline{c_1 f'(\psi(\overline{\xi}))} d\xi - \lambda_j f(z) \right], \tag{3.5}$$

$i, j = 1, 2, \ i \neq j$.

Differentiating the above equation for $i = 1$, we get

$$(P_1 f)'(z) = \frac{1}{\lambda_1 - \lambda_2} \left[\overline{c_1 f'(\psi(\overline{z}))} - \lambda_2 f'(z) \right].$$

Taking $z = 0$ in Equation (3.5), we obtain

$$P_1 f(0) = \frac{1}{\lambda_1 - \lambda_2} \left[c_0 f(0) - \lambda_2 f(0) \right].$$

Using all these information in the identity $TP_1 f(z) = \lambda_1 P_1 f(z)$,

$$c_0 (P_1 f)(0) + \int_{[0,z]} \overline{c_1 (P_1 f)'(\psi(\overline{\xi}))} d\xi = \frac{\lambda_1}{\lambda_1 - \lambda_2} \left[c_0 f(0) + \int_{[0,z]} \overline{c_1 f'(\psi(\overline{\xi}))} d\xi - \lambda_2 f(z) \right].$$

or

$$\begin{aligned}&\frac{c_0}{\lambda_1 - \lambda_2} [c_0 f(0) - \lambda_2 f(0)] + \frac{\overline{c_1}}{\overline{\lambda_1} - \overline{\lambda_2}} \int_{[0,z]} \left[\overline{c_1 f'(\psi(\overline{\psi(\overline{\xi})}))} - \lambda_2 f'(\psi(\overline{\xi})) \right] d\xi \\ &\qquad = \frac{\lambda_1}{\lambda_1 - \lambda_2} \left[c_0 f(0) + \int_{[0,z]} \overline{c_1 f'(\psi(\overline{\xi}))} d\xi - \lambda_2 f(z) \right].\end{aligned}$$

or

$$\frac{1}{\lambda_1-\lambda_2}[(c_0^2-c_0\lambda_1-c_0\lambda_2)f(0)+\lambda_1\lambda_2 f(z)]+\frac{1}{\overline{\lambda_1}-\overline{\lambda_2}}\int_{[0,z]} f'(\psi(\overline{\psi(\overline{\xi})}))d\xi=0. \tag{3.6}$$

Choosing $f=1$, we get

$$c_0^2-c_0\lambda_1-c_0\lambda_2+\lambda_1\lambda_2=0 \implies c_0=\lambda_1,\lambda_2.$$

Differentiating Equation (3.6), we obtain

$$\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}f'(z)+\frac{1}{\overline{\lambda_1}-\overline{\lambda_2}}f'(\psi(\overline{\psi(\overline{z})}))=0.$$

This implies that $\psi(\overline{\psi(\overline{z})})=z$ for all $z\in\mathbb{D}$.

We started the proof by taking $\lambda_1P_1+\lambda_2P_2$ to be an isometry T of form (III). We found out that $\psi(\overline{\psi(\overline{z})})=z$ for all $z\in\mathbb{D}$, which ensures that $P_i=\frac{T-\lambda_jI}{\lambda_i-\lambda_j}$ is indeed an idempotent map. It follows that the identity $\alpha P_1+\beta P_2=T$ holds true without any restrictions on α,β. Thus, the collection $\mathcal{C}$ forms a family of bi-circular idempotents, and this completes the proof.

□

Theorem 3.4. *If $\mathcal{C}=\{P_1,P_2\}$ is a family of generalized bi-circular idempotents on $\mathcal{X}$ corresponding to an isometry T of the form (IV), then it is a family of bi-circular idempotents. Moreover, $\psi(\overline{\psi(\overline{z})})=z$ for all $z\in\mathbb{D}$.*

Proof. Let $\mathcal{C}=\{P_1,P_2\}$ be a family of generalized bi-circular idempotents on $\mathcal{X}$ corresponding to an isometry T of form (IV). Then $T=\lambda_1P_1+\lambda_2P_2$, where $\lambda_1,\lambda_2\in\mathbb{T}$. Proposition 1.6 implies that $T(0)=0$. Thus, T takes the form

$$Tf(z)=\overline{c_0f(0)}+\int_{[0,z]}\overline{c_1f'(\psi(\overline{\xi}))}d\xi,\ \forall\ f\in\mathcal{X} \text{ and } z\in\mathbb{D}.$$

Further,

$$P_if(z)=\frac{Tf(z)-\lambda_jf(z)}{\lambda_i-\lambda_j}=\frac{1}{\lambda_1-\lambda_2}\left[\overline{c_0f(0)}+\int_{[0,z]}\overline{c_1f'(\psi(\overline{\xi}))}d\xi-\lambda_jf(z)\right],$$

$i,j=1,2,\ i\neq j$.

This implies that

$$(P_1f)'(z)=\frac{1}{\lambda_1-\lambda_2}\left[\overline{c_1f'(\psi(\overline{z}))}-\lambda_2f'(z)\right],$$

and

$$P_1f(0)=\frac{1}{\lambda_1-\lambda_2}\left[\overline{c_0f(0)}-\lambda_2f(0)\right].$$

$TP_1f(z)=\lambda_1P_1f(z)\implies$

$$\overline{c_0(P_1f)(0)} + \int_{[0,z]} \overline{c_1(P_1f)'(\psi(\overline{\xi}))d\xi}$$
$$= \frac{\lambda_1}{\lambda_1 - \lambda_2}\left[\overline{c_0f(0)} + \int_{[0,z]} \overline{c_1f'(\psi(\overline{\xi}))d\xi} - \lambda_2 f(z)\right].$$

It follows that

$$\frac{\overline{c_0}}{\overline{\lambda_1} - \overline{\lambda_2}}\left[\overline{\overline{c_0f(0)} - \lambda_2 f(0)}\right] + \frac{\overline{c_1}}{\overline{\lambda_1} - \overline{\lambda_2}}\int_{[0,z]}\left[\overline{\overline{c_1f'(\psi(\overline{\psi(\overline{\xi})}))d\xi} - \lambda_2 f'(\psi(\overline{\xi}))}\right] d\xi$$
$$= \frac{\lambda_1}{\lambda_1 - \lambda_2}\left[\overline{c_0f(0)} + \int_{[0,z]} \overline{c_1f'(\psi(\overline{\xi}))d\xi} - \lambda_2 f(z)\right],$$

or

$$\frac{1}{\overline{\lambda_1} - \overline{\lambda_2}} f(0) + \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} f(z) + \frac{1}{\overline{\lambda_1} - \overline{\lambda_2}} \int_{[0,z]} f'(\psi(\overline{\psi(\overline{\xi})}))d\xi = 0.$$

Now, differentiating the above equation, we get

$$\frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} f'(z) + \frac{1}{\overline{\lambda_1} - \overline{\lambda_2}} f'(\psi(\overline{\psi(\overline{z})})) = 0.$$

This implies that $\psi(\overline{\psi(\overline{z})}) = z$ for all $z \in \mathbb{D}$.

Using the same arguments at the end of the proof of Theorem 3.3, we can conclude that the collection $\mathcal{C}$ forms a family of bi-circular idempotents. This completes the proof. □

We end this paper with the following remark.

Remark 3.5. *Although Corollary 1.7 says that any GBI is real linear, it is clear from the structures of GBI obtained in the proofs of Theorems 3.2, 3.3, and 3.4 that they are nonlinear maps.*

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