APPROXIMATE LOCAL ISOMETRIES ON $C^{(2)}[0,1]$

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Abstract. Approximate local isometries on a Banach space X are bounded linear operators which are expressed pointwise as a (pointwise) limit of a sequence (which may change from point to point) of surjective linear isometries on X. Approximate 2-local isometries are maps on X (not necessarily linear or continuous) which, for every pair of points in X, are expressed as the 2-pointwise limit of a sequence (depending of the pair of points) of surjective linear isometries on X. In this paper, we give complete description of approximate local isometries on $C^{(2)}[0,1]$, the Banach space of 2-times continuously differentiable functions on $[0,1]$ equipped with the norm

$$\|f\|_\sigma = |f(0)| + |f'(0)| + \sup_{x\in[0,1]} |f''(x)|.$$

We also prove that every approximate 2-local isometry is an approximate local isometry.

1. Introduction

Let X be a Banach space, and $B(X)$ be the algebra of all bounded linear operators on X. We denote by $\mathcal{G}(X)$, the group of all surjective linear isometries on X. A map $T \in B(X)$ is said to be a local isometry if for every $x \in X$, there is $T_x \in \mathcal{G}(X)$ such that $T(x) = T_x(x)$. We say that T interpolates $\mathcal{G}(X)$ or T is locally in $\mathcal{G}(X)$. The problem here is to see whether this local property determine the class $\mathcal{G}(X)$ completely. In other words, does $T \in \mathcal{G}(X)$? Finding a global conclusion from a local condition is an important problem in Mathematics. Kadison, Larson and Sourour [15, 16, 17] have initiated such investigations. We note that one can state the above problem in a more general context by taking X to be an algebra, and $\mathcal{S}$ to be a collection of its linear transformations (for example, automorphism or derivations). A linear map $\phi : X \to X$ is said to be a local map or locally in $\mathcal{S}$ if for every $x \in X$, there is $\phi_x \in \mathcal{S}$ such that $\phi(x) = \phi_x(x)$. The monograph [23] by Molnár is a pertinent reference for the importance as well as applications of studying such problems.

The study of isometries on Banach spaces plays an important role to understand its structure and geometry. Characterization of local isometries on different Banach spaces has been studied extensively in the last decades. In this paper, we contribute to such a study but in a more general context.

We have defined local isometries or local maps to be linear. It is a natural question to extend this problem to nonlinear maps. We observe here that if we omit linearity from T,

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then the condition that T belongs locally to $\mathcal{G}(X)$ or any class $\mathcal{S} \subset B(X)$ alone may not give any reasonable results. For example, every map on the reals is locally linear. Indeed, given $F : \mathbb{R} \to \mathbb{R}$, for every real number t_0, we have that $y - F(t_0) = m(t - t_0)$ is equal to $F(t_0)$, thus it is locally linear. Further, if F is differentiable, we have $y - F(t_0) = F'(t_0)(t - t_0)$.

So, without linearity one needs to strengthen the local condition. This lead to the idea of 2-locality by Šemrl [24]. A map $T : X \to X$, not necessarily linear or bounded, is called a 2-local isometry if for every pair of points $x, y \in X$, there is $T_{x,y} \in \mathcal{G}(X)$ such that

$$T(x) = T_{x,y}(x) \text{ and } T(y) = T_{x,y}(y).$$

A more general way of looking at the above two problems, i.e., to see whether (1)-local or 2-local isometries belong to $\mathcal{G}(X)$, is in the sense of Loginov and Šulman [19]. Consider the following definitions.

Definition 1.1. (1) *A map $T \in B(X)$ is called an approximate local isometry if for every $x \in X$, there is a sequence $T_n^x \in \mathcal{G}(X)$ such that $T_n^x(x) \to T(x)$.*

(2) *A map $T : X \to X$, not necessarily linear or bounded, is called an approximate 2-local isometry if for every pair of points $x, y \in X$, there is a sequence $T_n^{x,y} \in \mathcal{G}(X)$ such that*

$$T_n^{x,y}(x) \to T(x) \text{ and } T_n^{x,y}(y) \to T(y).$$

(3) *The isometry group $\mathcal{G}(X)$ is called topologically (2-topologically) reflexive if every approximate local (2-local) isometry is a surjective isometry.*

It was proved by Molnár [20] that for an infinite dimensional separable Hilbert space H, the isometry group of $B(H)$ is topologically reflexive. The same result was extended to the isometry group of $\ell_\infty(\mathbb{N}, B(H))$ in [21], Hardy spaces, Banach algebras and Fréchet algebras of holomorphic functions in [4], and different classes of analytic functions (Novinger-Oberlin spaces, Kolaski spaces, the Ida-Mochizuki spaces and the Smirnov Class) in [3].

The study of approximate 2-local isometries was initiated very recently, see [5, 6, 7, 8, 9] and [10]. In [10], Jiménez-Vargas and Miura established the topological as well as 2-topological reflexivity of the isometry group of $S^p(\mathbb{D})$, the space of holomorphic functions f on the open unit disc $\mathbb{D}$ such that f' belongs to the Hardy space $H^p(\mathbb{D})$, where $1 \leq p \leq \infty$. For be a compact subset X of $\mathbb{R}$, Hosseini and Jiménez-Vargas in [7] described the structure of approximate local isometries of $C^{(n)}(X)$, the space of all n-times continuously differentiable functions on X with the norm $||f|| = \max_{x \in X} \left(\sum_{k=0}^{n} \frac{|f^{(k)}(x)|}{k!} \right)$. Moreover, if X is a compact interval in $\mathbb{R}$, then the isometry group of $C^{(n)}(X)$ is 2-topologically reflexivity. Furthermore, they studied similar problems for $AC(X)$, the space of all absolutely continuous functions on X, and $AC^p(X)$, the space of all absolutely continuous functions f on X such that $f' \in L^p(X)$, where $p \geq 1$, see [8] and [9]. Motivated by these results, in this paper, we find the structure of approximate local isometries on $C^{(2)}[0,1]$, equipped with the norm

$$\|f\|_\sigma = |f(0)| + |f'(0)| + \sup_{x \in [0,1]} |f''(x)|.$$

We also proof that any approximate 2-local isometry of $C^{(2)}[0,1]$ is an approximate local isometry. The main idea of [7], see also [8, 9, 10], is the application of spherical versions of Gleason-Kahane-Żelazko and Kowalski-Slodkowski theorems obtained in [18]. These two results apply to unital Banach algebras. In our case, the space $C^{(2)}[0,1]$ is not a Banach algebra with respect to the norm $\|\cdot\|_\sigma$. Indeed, consider $f(x) = g(x) = x$, $x \in [0,1]$. Then $||f||_\sigma = ||g||_\sigma = 1$, but $||fg||_\sigma = 2$. We follow a similar line of arguments presented in [7] but our approach to apply the spherical versions of Gleason-Kahane-Żelazko theorem and Kowalski-Slodkowski theorem is quite different.

2. Preliminaries and basic results

In this section, we recall some facts and results which will be used in this paper. Before proceeding, let us address a question which may puzzle a first-time reader. The concept of 2-locality was defined for nonlinear maps. One might wonder that why did we stop at 2? What about 3-local or n-local maps? Moreover, what happens to linear 2-local maps? We answer these questions in the coming remarks.

Remark 2.1. *Let ϕ be a 3-local homomorphism on an algebra $\mathcal{A}$. We will show that ϕ is a homomorphism, i.e., it is linear and multiplicative.*

Let $a, b \in \mathcal{A}$ and λ be any scalar. Let $c = \lambda a$, $d = a + b$ and $e = ab$.

Since any 3-local map is also 2-local, for $a, c \in \mathcal{A}$, there exists a homomorphism $\phi_{a,c}$ on $\mathcal{A}$ such that

$$\phi(a) = \phi_{(a,c)}(a) \text{ and } \phi(\lambda a) = \phi(c) = \phi_{(a,c)}(c) = \phi_{(a,c)}(\lambda a) = \lambda\phi_{(a,c)}(a) = \lambda\phi(a).$$

Moreover, there exists a homomorphism $\phi_{a,b,d}$ on $\mathcal{A}$ such that $\phi(a) = \phi_{(a,b,d)}(a)$, $\phi(b) = \phi_{(a,b,d)}(b)$ and

$$\phi(a+b) = \phi(d) = \phi_{(a,b,d)}(d) = \phi_{(a,b,d)}(a+b) = \phi_{(a,b,d)}(a) + \phi_{(a,b,d)}(b) = \phi(a) + \phi(b).$$

Thus, ϕ is linear. Furthermore, there exists a homomorphism $\phi_{a,b,e}$ on $\mathcal{A}$ such that $\phi(a) = \phi_{(a,b,e)}(a)$, $\phi(b) = \phi_{(a,b,e)}(b)$ and

$$\phi(ab) = \phi(e) = \phi_{(a,b,e)}(e) = \phi_{(a,b,e)}(ab) = \phi_{(a,b,e)}(a)\phi_{(a,b,e)}(b) = \phi(a)\phi(b).$$

Therefore, ϕ is multiplicative. So, a 3-local homomorphism is indeed an algebra homomorphism. So, there is no point of going beyond 2-local maps.

Remark 2.2. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. We know that $\mathcal{G}(\mathcal{H})$ is not algebraically reflexive. The same reasoning could be applied to show that $\mathcal{G}(\mathcal{H})$ fails to be 2-algebraically reflexive as well. Let S be an into isometry on $\mathcal{H}$. We claim that S is a 2-local isometry. Let $x, y \in \mathcal{H}$. We may assume $\{x, y\}$ to be an orthonormal set. We complete this set in order to get an orthonormal basis $\mathcal{B} = \{u_1, u_2, \ldots, u_n, \ldots\}$ with $u_1 = x$ and $u_2 = y$ (here, we are assuming $\mathcal{H}$ to be separable). Similarly, $\{S(x), S(y)\}$ is an orthonormal set which can* *be extended to an orthonormal basis* $\mathcal{B}_1 = \{w_1, w_2, \dots w_n, \dots\}$ *with* $w_1 = S(x)$ *and* $w_2 = S(y)$. *Define the map* $T(u_n) = w_n$. *Clearly,* $T \in \mathcal{G}(\mathcal{H})$.

For $f \in C[0,1]$, the integral operator $S : C[0,1] \to C^{(1)}[0,1]$ is defined as

$$(Sf)(x) = \int_0^x f(t)dt, \ \forall\ x \in [0,1].$$

Then $S^2 : C[0,1] \to C^{(r)}[0,1]$ is given by

$$(S^2 f)(x) = \int_0^x \int_0^t f(s)dsdt, \ \forall\ x \in [0,1].$$

The structure of surjective linear isometries on $C^{(2)}[0,1]$ is described by Koshimizu in [13]. Koshimizu's result actually describes the isometries on $C^{(r)}[0,1]$, $r \geq 1$, but we state it only for $r = 2$.

Theorem 2.3. [13, Theorem 1.1] *Let* T *be a linear operator form* $C^{(2)}[0,1]$ *onto itself. Then* T *is an isometry if and only if there exist a homomorphism* $\phi : [0,1] \to [0,1]$, *a continuous function* $\omega : [0,1] \to \mathbb{T}$, *a permutation* $\{\tau(0), \tau(1)\}$ *of* $\{0,1\}$ *and unimodular constants* λ, μ *such that*

$$Tf(x) = \lambda f^{\tau(0)}(0) + \mu f^{\tau(1)}(0)x + (S^2(\omega(f'') \circ \phi)))(x), \tag{2.1}$$

Since $\{0,1\}$ has two permutations, the above theorem implies that there are two forms of surjective linear isometries of $C^{(2)}[0,1]$.

For a unital Banach algebra $\mathcal{A}$ and $a \in \mathcal{A}$, let $\sigma(a)$ denotes the spectrum of a. Our results strongly rely on the spherical versions of Gleason-Kahane-Żelazko and Kowalski-Slodkowski theorems which are stated below.

Theorem 2.4. [18, Proposition 2.2] *Let* $F : \mathcal{A} \to \mathbb{C}$ *be a continuous linear functional, where* $\mathcal{A}$ *is a unital complex Banach algebra. Suppose that* $F(a) \in \mathbb{T}\sigma(a)$, *for every* $a \in \mathcal{A}$. *Then the mapping* $F(1)F$ *is multiplicative.*

Theorem 2.5. [18, Proposition 3.2] *Let* $\mathcal{A}$ *be a unital complex Banach algebra, and let* $F : \mathcal{A} \to \mathbb{C}$ *be a mapping satisfying the following properties:*

(1) F *is* 1*-homogeneous.*
(2) $F(x) - F(y) \in \mathbb{T}\sigma(x - y)$ *for every* $x, y \in \mathcal{A}$.

Then F *is linear, and there exists* $\lambda_0 \in \mathbb{T}$ *such that* $\lambda_0 F$ *is multiplicative.*

A few remarks are in order.

Remark 2.6. (1) *Any approximate local isometry* T *is an isometry. Indeed, for any* $x \in X$, *there exist* $T_n^x \in \mathcal{G}(X)$ *such that* $T_n^x(x) \to T(x)$. *This implies that* $||T_n^x(x)|| \to ||T(x)||$ *or* $||x|| = ||T(x)||$.

(2) *For any* $f \in C^{(2)}[0,1]$, *we have* $|f(0)| \leq ||f||_\sigma$, $|f'(0)| \leq ||f||_\sigma$ *and* $||f''||_\infty \leq ||f||_\sigma$. *It follows that if* $f_n \xrightarrow{||\cdot||_\sigma} f$, *then* $f_n(0) \to f(0)$, $f_n'(0) \to f'(0)$ *and* $f_n'' \xrightarrow{||\cdot||_\infty} f''$.

(3) *For any* $T \in B(C^{(2)}[0,1])$ *and* $f \in C^{(2)}[0,1]$, *it is easy to verify that*

$$Tf(x) = Tf(0) + (Tf)'(0)x + (S^2((Tf)''))(x). \tag{2.2}$$

(4) *If* T *is an approximate local isometry on* $C^{(2)}[0,1]$, *then for any* $f \in C^{(2)}[0,1]$ *there is a sequence* $T_n^f \in \mathcal{G}(C^{(2)}[0,1])$ *such that* $T_n^f f \to Tf$. *Theorem 2.3 implies the existence of sequences of*

a) homomorphisms $\phi_n^f : [0,1] \to [0,1]$, *b) continuous functions* $\omega_n^f : [0,1] \to \mathbb{T}$,

c) permutations τ_n^f *of* $\{0,1\}$, *and* *d) unimodular constants* λ_n^f, μ_n^f

such that

$$Tf(x) = \lim_{n\to\infty} \left[\lambda_n^f f^{\tau_n^f(0)}(0) + \mu_n^f f^{\tau_n^f(1)}(0)x + (S^2(\omega_n^f(f'' \circ \phi_n^f)))(x)\right].$$

(5) *If* T *is an approximate 2-local isometry, then for any pair of functions* $f, g \in C^{(2)}[0,1]$, *there is a sequence* $T_n^{f,g} \in \mathcal{G}(C^{(2)}[0,1])$ *such that* $T_n^{f,g} f \to Tf$ *and* $T_n^{f,g} g \to Tg$. *An application of Theorem 2.3 again gives us sequences* $\phi_n^{f,g}$, $\omega_n^{f,g}$, $\tau_n^{f,g}$, $\lambda_n^{f,g}$ *and* $\mu_n^{f,g}$ *similar to 4(a), 4(b), 4(c) and 4(d) above, but depending on both* f *and* g *such that*

$$Tf(x) = \lim_{n\to\infty} \left[\lambda_n^{f,g} f^{\tau_n^{f,g}(0)}(0) + \mu_n^{f,g} f^{\tau_n^f(1)}(0)x + (S^2(\omega_n^{f,g}(f'' \circ \phi_n^{f,g})))(x)\right],$$

and

$$Tg(x) = \lim_{n\to\infty} \left[\lambda_n^{f,g} g^{\tau_n^{f,g}(0)}(0) + \mu_n^{f,g} g^{\tau_n^f(1)}(0)x + (S^2(\omega_n^{f,g}(g'' \circ \phi_n^{f,g})))(x)\right].$$

(6) *The above two facts, i.e., (4) and (5), will be used again and again in the whole paper. To avoid repetition, we shall write the expression of the approximate local (2-local) isometry without mentioning details of the sequences. It will be understood from the superscript of sequences that they are associated with the function(s)* f *(*f *and* g*) and the sequence of isometries* T_n^f *(*$T_n^{f,g}$*).*

The following example shows that $\mathcal{G}(C^{(2)}[0,1])$ is not topologically reflexive. Thus, it makes sense to describe the structure of approximate local isometries on $C^{(2)}[0,1]$ explicitly.

Example 2.7. *For* $n \in \mathbb{N}$, *define the functions* ϕ_n *and* ϕ *on* $[0,1]$ *as*

$$\phi_n(x) = \begin{cases} \frac{2}{n+1}x, & 0 \le x < \frac{n+1}{2n}, \\ 2x-1, & \frac{n+1}{2n} \le x \le 1. \end{cases} \tag{2.3}$$

and

$$\phi(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2}, \\ 2x-1, & \frac{1}{2} \le x \le 1. \end{cases} \tag{2.4}$$

For $n \in \mathbb{N}$, *we also define* $T_n, T : C^{(2)}[0,1] \to C^{(2)}[0,1]$ *as*

$$T_n f(x) = f^{\tau(0)}(0) + f^{\tau(1)}(0)x + (S^2(f'' \circ \phi_n))(x),$$

and

$$Tf(x) = f^{\tau(0)}(0) + f^{\tau(1)}(0)x + (S^2((f'' \circ \phi))(x)$$

for some permutation $\{\tau(0), \tau(1)\}$ *of* $\{0,1\}$*. Clearly,* $T_n \in \mathcal{G}(C^{(2)}[0,1])$ *but* $T \notin \mathcal{G}(C^{(2)}[0,1])$*. We claim that* $\lim_{n\to\infty} T_n f = Tf$*, for every* $f \in C^{(2)}[0,1]$*. Indeed,*

$$\begin{aligned}
\|T_n f - Tf\|_\sigma &= |(T_n f - Tf)(0)| + |(T_n f - Tf)'(0)| + \sup_{x\in[0,1]} |(T_n f - Tf)''(x)| \\
&= |T_n f(0) - Tf(0)| + |(T_n f)'(0) - (Tf)'(0)| + \sup_{x\in[0,1]} |(T_n f)''(x) - (Tf)''(x)| \\
&= \sup_{x\in[0,1]} |(f'' \circ \phi_n)(x) - (f'' \circ \phi))(x)| \\
&= |(f'' \circ \phi_n)(x_0) - (f'' \circ \phi))(x_0)| \text{ for some } x_0 \in [0,1] \\
&= |f''(\phi_n(x_0) - f''(\phi(x_0)| \\
&\to 0 \text{ as } n \to \infty, \text{ since } \lim_{n\to\infty} \phi_n = \phi.
\end{aligned}$$

3. Structure of approximate local isometries on $C^{(2)}[0,1]$

In this section we state and prove the main result of this paper. It gives the structure of approximate local isometries on $C^{(2)}[0,1]$. For the sake of clarity, the proof is divided into several steps.

Theorem 3.1. *Every approximate local isometry* T *of* $C^{(2)}[0,1]$ *is linear isometry of the form*

$$Tf(x) = \lambda f^{\tau(0)}(0) + \mu f^{\tau(1)}(0)x + (S^2(\omega(f'' \circ \phi)))(x),$$

where, $\phi : [0,1] \to [0,1]$ *is a surjective continuous map,* $\omega : [0,1] \to \mathbb{T}$ *is a continuous function,* $\{\tau(0), \tau(1)\}$ *is a permutation of* $\{0,1\}$ *and* λ, μ *are unimodular constants. Moreover,* T *is surjective if and only if* ϕ *is injective.*

Proof. Let T be an approximate local isometry on $C^{(2)}[0,1]$. For $f \in C^{(2)}[0,1]$, there exists a sequence $T_n^f \in \mathcal{G}(C^{(2)}[0,1])$ such that $T_n^f f \to Tf$. Since there are two permutations of $\{0,1\}$, without loss of generality, we may assume that all, except finitely many, terms of the sequence $(T_n f)$ can be expressed with a single permutation τ_f. Thus, there exist a sequence of homomorphism $\phi_n^f : [0,1] \to [0,1]$, a sequence of continuous functions $\omega_n^f : [0,1] \to \mathbb{T}$, a permutation τ^f of $\{0,1\}$ and sequence of unimodular constants λ_n^f, μ_n^f such that

$$Tf(x) = \lim_{n\to\infty} \Big(\lambda_n^f f^{\tau^f(0)}(0) + \mu_n^f f^{\tau^f(1)}(0)x + (S^2(\omega_n^f(f'' \circ \phi_n^f)))(x) \Big).$$

From point 2 of Remark 2.6, we have the following:

$$T_n^f f(0) \to Tf(0), \quad (T_n^f f)'(0) \to (Tf)'(0), \quad \text{and} \quad (T_n^f f)''(0) \to (Tf)''(0)$$

for every $f \in C^{(2)}[0,1]$.

For $f = 1$, we obtain:

$$T1(0) = \lim_{n\to\infty} T_n^1 1(0) = \lim_{n\to\infty} \Big(\lambda_n^1 1^{\tau^1(0)}(0) \Big) \tag{3.1}$$

and

$$(T1)'(0)=\lim_{n\to\infty}(T_n^1 1)'(0)=\lim_{n\to\infty}\left(\mu_n^1 1^{\tau^1(1)}(0)\right). \tag{3.2}$$

Similarly, by taking $f=\mathrm{id}$, we get:

$$T\mathrm{id}(0)=\lim_{n\to\infty}T_n^{\mathrm{id}}\mathrm{id}(0)=\lim_{n\to\infty}\left(\lambda_n^{\mathrm{id}}\mathrm{id}^{\tau^{id}(0)}(0)\right) \tag{3.3}$$

and

$$(T\mathrm{id})'(0)=\lim_{n\to\infty}(T_n^{\mathrm{id}}\mathrm{id})'(0)=\lim_{n\to\infty}\left(\mu_n^{\mathrm{id}}\mathrm{id}^{\tau^{\mathrm{id}}(1)}(0)\right). \tag{3.4}$$

From equations 3.1 and 3.2, we obtain

$$|T1(0)|=1^{\tau^1(0)}(0)\quad\text{and}\quad|(T1)'(0)|=1^{\tau^1(1)}(0),$$

respectively. Therefore, we have

$$T1(0)=\lambda 1^{\tau^1(0)}(0)\quad\text{and}\quad(T1)'(0)=\lambda' 1^{\tau^1(1)}(0), \tag{3.5}$$

for some $\lambda,\lambda'\in\mathbb{T}$. Similarly, from equations 3.3 and 3.4, we get

$$T\mathrm{id}(0)=\mu\mathrm{id}^{\tau^{\mathrm{id}}(0)}(0)\quad\text{and}\quad(T\mathrm{id})'(0)=\mu'\mathrm{id}^{\tau^{\mathrm{id}}(1)}(0), \tag{3.6}$$

for some $\mu,\mu'\in\mathbb{T}$.

For $f\in C^{(2)}[0,1]$, define a function $h=f-f(0)1-f'(0)\mathrm{id}$. It is clear that $h\in C^{(2)}[0,1]$. Since T is an approximate local isometry, therefore we have

$$Th(x)=\lim_{n\to\infty}\left(\lambda_n^h h^{\tau^h(0)}(0)+\mu_n^h h^{\tau^h(1)}(0)x+(S^2(\omega_n^h(h''\circ\phi_n^h)))(x)\right).$$

At $x=0$ in the above equation, we have

$$Th(0)=\lim_{n\to\infty}\left(\lambda_n^h h^{\tau^h(0)}(0)\right)=\lim_{n\to\infty}\left(\lambda_n^h\Big(f^{\tau^h(0)}(0)-f(0)1^{\tau^h(0)}(0)-f'(0)\mathrm{id}^{\tau^h(0)}(0)\Big)\right). \tag{3.7}$$

For both permutations τ^h on $\{0,1\}$ in Equation (3.7), we obtain $Th(0)=0$. This gives the following relation:

$$Tf(0)=f(0)T1(0)+f'(0)T\mathrm{id}(0).$$

Using Equations 3.5 and 3.6, along with the above equation, we derive

$$Tf(0)=\lambda 1^{\tau^1(0)}(0)f(0)+\mu\ \mathrm{id}^{\tau^{\mathrm{id}}(0)}(0)f'(0). \tag{3.8}$$

Next, we compute $(Th)'$ at the point $x=0$. We obtain

$$(Th)'(0)=\lim_{n\to\infty}\left(\mu_n^h h^{\tau^h(1)}(0)\right)=\lim_{n\to\infty}\left(\mu_n^h\left(f^{\tau^h(1)}(0)-f(0)1^{\tau^h(1)}(0)-f'(0)\,\mathrm{id}^{\tau^h(1)}(0)\right)\right). \tag{3.9}$$

Using both permutations τ^h on $\{0,1\}$ in Equation (3.9), we arrive at the condition $(Th)'(0)=0$. This leads to the following relation:

$$(Tf)'(0)=f(0)(T1)'(0)+f'(0)(T\,\mathrm{id})'(0).$$

Utilizing Equations (3.5) and (3.6), along with the equation above, we derive

$$(Tf)'(0)=\lambda' 1^{\tau^1(1)}(0)f(0)+\mu'\ \mathrm{id}^{\tau^{\mathrm{id}}(1)}(0)f'(0). \tag{3.10}$$

This result implies that there are four possible values for $Tf(0)$ and $(Tf)'(0)$, each determined by the permutations τ^1 and τ^{id}, as outlined below:

(1) $\tau^1 = I$ and $\tau^{\text{id}} = I$, where I is the identity permutation. From equation 3.8 and 3.10, it follows that $Tf(0) = \lambda f(0)$, $(Tf)'(0) = \mu' f'(0)$.

(2) $\tau^1 = I$ and $\tau^{\text{id}} = (0\ 1)$. As a result, equation 3.8 and 3.10, gives $Tf(0) = \lambda f(0) + \mu f'(0)$, $(Tf)'(0) = 0$.

(3) $\tau^1 = (0\ 1)$ and $\tau^{\text{id}} = I$. According to equation 3.8 and 3.10, we find $Tf(0) = 0$, $(Tf)'(0) = \lambda' f(0) + \mu' f'(0)$.

(4) $\tau^1 = (0\ 1)$ and $\tau^{\text{id}} = (0\ 1)$. Consequently, equation 3.8 and 3.10, implies $Tf(0) = \mu f'(0)$, $(Tf)'(0) = \lambda' f(0)$.

We complete the proof in several steps.

Step 1. For $x \in [0,1]$ and $g = \frac{(\text{id})^2}{2}$, define $\omega(x) = (Tg)''(x)$. Then there exist ω_n^g, ϕ_n^g such that $\omega_n^g(1 \circ \phi_n^g) \to \omega$ (in sup norm). This implies that $\omega_n^g(x) \to \omega(x)$ and hence, $|\omega(x)| = 1$.

Step 2. For $x \in [0,1]$, define the map $J_x : C[0,1] \to \mathbb{C}$ by $J_x(f) = \overline{\omega(x)}(T(S^2 f))''(x)$. We claim that J_x is a unital multiplicative linear functional.

It is clear that J_x is a linear functional. Moreover,

$$J_x(1) = \overline{\omega(x)}(T(S^2 1))''(x) = \overline{\omega(x)}(T(\frac{(\text{id})^2}{2}))''(x) = \overline{\omega(x)}\omega(x) = |\omega(x)|^2 = 1.$$

Therefore, J_x is unital.

For $x \in [0,1]$, define the map $T_x : C[0,1] \to \mathbb{C}$ by $T_x(f) = (T(S^2 f))''(x)$. Since T_x is linear, and for all $f \in C[0,1]$, we have

$$\begin{aligned}
|T_x(f)| &= |(T(S^2 f))''(x)| \\
&\le \|(T(S^2 f))''\|_\infty \\
&\le \|(T(S^2 f))\|_\sigma \\
&= \|S^2 f\|_\sigma \\
&= |(S^2 f)(0)| + |(S^2 f)'(0)| + \sup_{x \in [0,1]} |(S^2 f)''(x)| \\
&= \|f\|_\infty.
\end{aligned}$$

This implies that T_x is a continuous linear functional.

For any $f \in C[0,1]$ we have $S^2 f \in C^{(2)}[0,1]$, then there exist sequences $\phi_n^{S^2 f}$ and $\omega_n^{S^2 f}$ such that

$$(T(S^2 f))'' = \lim_{n \to \infty} \omega_n^{S^2 f}(f \circ \phi_n^{S^2 f}).$$

Hence,

$$T_x(f) = (T(S^2 f))''(x) = \lim_{n \to \infty} \omega_n^{S^2 f}(x) f(\phi_n^{S^2 f}(x)) \in \mathbb{T}\sigma(f).$$

Applying Theorem 2.4, we conclude that $J_x = \overline{T_x(1)} T_x$ is multiplicative.

Step 3. There exits a surjective continuous function $\phi : [0,1] \to [0,1]$ such that $(Tf)''(x) = \omega(x) f''(\phi(x))$ for all $f \in C^{(2)}[0,1], x \in [0,1]$.

The map $J : C[0,1] \to C[0,1]$ by $Jf(x) = J_x(f)$ is a unital algebra homomorphism. It is known that such a map is of the form $f \mapsto f \circ \phi$ for some continuous map $\phi : [0,1] \to [0,1]$. It follows that $(T(S^2 f))''(x) = \omega(x) f(\phi(x))$ for $f \in C[0,1]$, and $x \in [0,1]$).

Step 4. For $f \in C^{(2)}[0,1]$, let $g = S^2(f'')$. Then $g \in C^{(2)}([0,1])$ and $(Tf)'' - (Tg)'' = (Tf - Tg)'' = (T(f-g))''$. As T is an approximate local isometry, there exist sequences ω_n^{f-g} and ϕ_n^{f-g} such that

$$(T(f-g))'' = \omega_n^{f-g}(f-g)'' \circ \phi_n^{(f-k)} = \omega_n^{f-g}(f'' - f'') \circ \phi_n^{f-g}) = 0.$$

Hence,

$$(Tf)'' = (Tg)'' = (T(S^2(f'')))'' = \omega(f'' \circ \phi). \tag{3.11}$$

Putting the values of $Tf(0)$ and $(Tf)'(0)$ obtained in (1)-(4), and the value of $(Tf)''$ from Equation 3.11, in Equation (2.2) we conclude that T is of the desired form.

Step 5. It remains to show that ϕ is surjective. If not, then there exists $t \in [0,1] \setminus \phi([0,1])$. We choose a function $f \in C[0,1]$ such that $f(t) = 1$ and $f|_{\phi([0,1])} = 0$. This implies that

$$\begin{aligned} T(S^2 f)(x) &= |(S^2 f)(0)| + |(S^2 f)'(0)| + (S^2(\omega(S^2 f)'' \circ \phi))(x) \\ &= (S^2(\omega(f \circ \phi)))(x) \\ &= 0 \end{aligned}$$

for all $x \in [0,1]$, and thus, $S^2 f = 0$ because T is injective. Since f is continuous, $f = 0$, which is a contradiction.

Step 6. T is surjective if and only if ϕ is injective.

From the previous step, we have

$$Tf(x) = \lambda f^{\tau(0)}(0) + \mu f^{\tau(1)}(0)x + (S^2(\omega(f'' \circ \phi)))(x).$$

For the direct part, we let $x_1, x_2 \in [0,1]$ such that $\phi(x_1) = \phi(x_2)$. Then

$$\overline{\omega(x_1)}(Tf)''(x_1) = f''(\phi(x_1)) = f''(\phi(x_2)) = \overline{\omega(x_2)}(Tf)''(x_2),$$

for all $f \in C^{(2)}[0,1]$.

Since T is surjective, we conclude

$$\overline{\omega(x_1)} g''(x_1) = \overline{\omega(x_2)} g''(x_2),$$

for all $g \in C^{(2)}[0,1]$.

Taking $g = x^3$ and applying modulus on both sides, we obtain $x_1 = x_2$. This implies that ϕ is injective.

Conversely, let $g \in C^{(2)}[0,1]$.

Case 1: If $\tau = I$, define

$$f(x) = \overline{\lambda} g(0) + \overline{\mu} g'(0)x + (S^2(\overline{\omega}(g'' \circ \phi^{-1})))(x).$$

Clearly, $f \in C^{(2)}[0,1]$. This implies that

$$\begin{aligned} Tf(x) &= \lambda f(0) + \mu f'(0)x + (S^2(\omega(f'' \circ \phi)))(x) \\ &= g(0) + g^{'}(0)x + (S^2(\omega(\overline{\omega}(g'' \circ \phi^{-1} \circ \phi))))(x) \\ &= g(0) + g^{'}(0)x + (S^2 g'')(x) \\ &= g(0) + g^{'}(0)x + g(x) - g(0) - g'(0)x \\ &= g(x). \end{aligned}$$

Case 2: If $\tau = (0\ 1)$, define

$$f(x) = \overline{\mu}g(0) + \overline{\lambda}g'(0)x + (S^2(\overline{\omega}(g'' \circ \phi^{-1})))(x).$$

Again, $f \in C^{(2)}[0,1]$. It follows that

$$\begin{aligned} Tf(x) &= \lambda f'(0) + \mu f(0)x + (S^2(\omega(f'' \circ \phi)))(x) \\ &= g'(0) + g(0)x + (S^2(\omega(\overline{\omega}(g'' \circ \phi^{-1} \circ \phi))))(x) \\ &= g(0) + g^{'}(0)x + (S^2 g'')(x) \\ &= g(0) + g^{'}(0)x + g(x) - g(0) - g'(0)x \\ &= g(x). \end{aligned}$$

Therefore, T is a surjective linear isometry on $C^{(2)}[0,1]$.

This completes the proof. □

We can apply Theorem 3.1 to give an alternate proof of [1, Theorem 3.1] which states that the group of surjective linear isometries on $C^{(2)}[0,1]$ is algebraically reflexive.

Corollary 3.2. *Every local isometry of* $C^{(2)}[0,1]$ *to* $C^{(2)}[0,1]$ *is a surjective.*

Proof. Let T be a local isometry of $C^{(2)}[0,1]$. Then, by Theorem 3.1, T is given by

$$Tf(x) = \lambda f^{\tau(0)}(0) + \mu f^{\tau(1)}(0)x + (S^2(\omega(f'' \circ \phi)))(x),$$

where, $\phi : [0,1] \to [0,1]$ is a surjective continuous map, $\omega : [0,1] \to \mathbb{T}$ is a continuous function, $\{\tau(0), \tau(1)\}$ is a permutation of $\{0,1\}$ and λ, μ are unimodular constants. We only need to prove that ϕ is injective.

Since T is a local isometry, for every $f \in C^{(2)}[0,1]$, there exists a surjective linear isometry T_f such that $Tf = T_f f$. Moreover, we have $(Tf)'' = (T_f f)''$. This implies that

$$\omega(x) f''(\phi(x)) = \omega_f(x) f''(\phi_f(x)), \quad \text{for all } x \in [0,1].$$

Taking $f(x) = x^3$ and modulus on both sides, we obtain $\phi(x) = \phi_f(x)$ for every $x \in [0,1]$. Since ϕ_f is injective, it follows that ϕ is also injective. □

4. Structure of approximate 2-local isometries on $C^{(2)}[0,1]$

In this section we describe the structure of approximate 2-local isometries on $C^{(2)}[0,1]$. Recall that those are maps defined on $C^{(2)}[0,1]$ satisfying the 2-local condition without any assumption of linearity or boundedness.

Theorem 4.1. *Every approximate 2-local isometry of $C^{(2)}[0,1]$ is an approximate local isometry.*

Proof. Let T be an approximate 2-local isometry on $C^{(2)}[0,1]$. For $x \in [0,1]$, we define a functional T_x on $C[0,1]$ by

$$T_x(f) = (T(S^2 f))''(x).$$

We claim that T_x is linear. Let $f \in C[0,1]$, $g = S^2 f$ and $h = S^2(\lambda f)$. Since T is an approximate 2-local isometry, we have

$$T_x(f) = (Tg)''(x) = \lim_{n\to\infty} \omega_n^{g,h}(x) f(\phi_n^{g,h}(x))$$

and

$$T_x(\lambda f) = (Th)''(x) = \lambda \lim_{n\to\infty} \omega_n^{g,h}(x) f(\phi_n^{g,h}(x)).$$

This implies that $T_x(\lambda f) = \lambda T_x(f)$ for every $f \in C[0,1]$ and scalar λ. Therefore, T_x is 1-homogeneous. Similarly, for $f, g \in C([0,1])$ we have

$$T_x(f) - T_x(g) = \lim_{n\to\infty} \omega_n^{S^2 f, S^2 g}(x)(f-g)(\phi_n^{S^2 f, S^2 g}(x)) \in \mathbb{T}\sigma(f-g).$$

Theorem 2.5 implies that T_x is linear. For $f \in C^{(2)}[0,1]$, let $g = S^2(f'')$. Then $g \in C^{(2)}[0,1]$. Moreover,

$$\begin{aligned}
(Tf)''(x) - (Tg)''(x) &= \lim_{n\to\infty} \Big(\omega_n^{f,g}(x)(f'')(\phi_n^{f,g}(x)) - \omega_n^{f,g}(x)(g'')(\phi_n^{f,g}(x)) \Big) \\
&= \lim_{n\to\infty} \Big(\omega_n^{f,g}(x)(f'')(\phi_n^{f,g}(x)) - \omega_n^{f,g}(x)(f'')(\phi_n^{f,g}(x)) \Big) \\
&= \lim_{n\to\infty} \Big(\omega_n^{f,g}(x)(f'' - f'')(\phi_n^{f,g}(x)) \Big) = 0.
\end{aligned}$$

Hence,

$$(Tf)''(x) = (Tg)''(x) = (T(S^2(f'')))''(x) = T_x(f''), \ \forall\, x \in [0,1], \ f \in C^{(2)}[0,1].$$

This implies that $D^2 \circ T$ is a linear map on $C^{(2)}[0,1]$.

Now, we calculate $Tf(0)$ and $(Tf)'(0)$. For the constant function 1 and for $f \in C^{(2)}[0,1]$, there exist a sequence $T_n^{1,f} \in \mathcal{G}(C^{(2)}[0,1])$ such that $T_n^{1,f} f \to Tf$ and $T_n^{1,f} 1 \to T1$. Without loss of generality, we may assume that all, except finitely many, terms of the sequence $(T_n^{1,f})$ can be expressed with a single permutation $\tau^{1,f}$. It follows that

$$T1(0) = \lim_{n\to\infty} T_n^{1,f} 1(0) = \lim_{n\to\infty} \Big(\lambda_n^{1,f} 1^{\tau^{1,f}(0)}(0) \Big), \tag{4.1}$$

$$(T1)'(0) = \lim_{n\to\infty} (T_n^{1,f} 1)'(0) = \lim_{n\to\infty} \Big(\mu_n^{1,f} 1^{\tau^{1,f}(1)}(0) \Big). \tag{4.2}$$

and

$$Tf(0)=\lim_{n\to\infty}T_n^{1,f}f(0)=\lim_{n\to\infty}\left(\lambda_n^{1,f}f^{\tau^{1,f}(0)}(0)\right), \tag{4.3}$$

$$(Tf)'(0)=\lim_{n\to\infty}(T_n^{1,f}f)'(0)=\lim_{n\to\infty}\left(\mu_n^{1,f}f^{\tau^{1,f}(1)}(0)\right). \tag{4.4}$$

This implies that there are two possible values for $Tf(0)$ and $(Tf)'(0)$, each determined by the permutations $\tau^{1,f}$, as follows:

(1) $\tau^{1,f}=I$. From Equations (4.1)-(4.4), we conclude that $Tf(0)=(T1)(0)f(0)$ and $(Tf)'(0)=f'(0)\lim_{n\to\infty}\mu_n^{1,f}$.
(2) $\tau^{1,f}=(0\ 1)$. As a result, Equations (4.1)-(4.4), give $Tf(0)=f'(0)\lim_{n\to\infty}\lambda_n^{1,f}$ and $(Tf)'(0)=(T1)'(0)f(0)$.

Similarly, by taking the functions id and any f, we get:

$$T\mathrm{id}(0)=\lim_{n\to\infty}T_n^{\mathrm{id},f}\mathrm{id}(0)=\lim_{n\to\infty}\left(\lambda_n^{\mathrm{id},f}\mathrm{id}^{\tau^{\mathrm{id},f}(0)}(0)\right), \tag{4.5}$$

$$(T\mathrm{id})'(0)=\lim_{n\to\infty}(T_n^{\mathrm{id},f}\mathrm{id})'(0)=\lim_{n\to\infty}\left(\mu_n^{\mathrm{id},f}\mathrm{id}^{\tau^{\mathrm{id},f}(1)}(0)\right). \tag{4.6}$$

and

$$Tf(0)=\lim_{n\to\infty}T_n^{\mathrm{id},f}f(0)=\lim_{n\to\infty}\left(\lambda_n^{\mathrm{id},f}f^{\tau^{\mathrm{id},f}(0)}(0)\right), \tag{4.7}$$

$$(Tf)'(0)=\lim_{n\to\infty}(T_n^{\mathrm{id},f}f)'(0)=\lim_{n\to\infty}\left(\mu_n^{\mathrm{id},f}f^{\tau^{\mathrm{id},f}(1)}(0)\right). \tag{4.8}$$

This implies that both $Tf(0)$ and $(Tf)'(0)$ are determined by the permutations $\tau^{1,f}$, with each having two possible values, as follows:

(1) $\tau^{\mathrm{id},f}=I$, where I is the identity permutation. From Equations (4.5)-(4.8), it follows that $Tf(0)=f(0)\lim_{n\to\infty}\lambda_n^{\mathrm{id},f}$ and $(Tf)'(0)=(T\mathrm{id})'(0)f'(0)$.
(2) $\tau^{\mathrm{id},f}=(0\ 1)$. As a result, Equations (4.5)-(4.8), give us $Tf(0)=(T\mathrm{id})(0)f'(0)$ and $(Tf)'(0)=f(0)\lim_{n\to\infty}\mu_n^{\mathrm{id},f}(0)$.

Therefore, for every $f\in C^{(2)}[0,1]$, we have four possible cases:

(1) $Tf(0)=(T1)(0)f(0)$, $(Tf)'(0)=f'(0)\lim_{n\to\infty}\mu_n^{1,f}$ and $Tf(0)=f(0)\lim_{n\to\infty}\lambda_n^{\mathrm{id},f}$, $(Tf)'(0)=(T\mathrm{id})'(0)f'(0)$.
(2) $Tf(0)=(T1)(0)f(0)$, $(Tf)'(0)=f'(0)\lim_{n\to\infty}\mu_n^{1,f}$, and $Tf(0)=(T\mathrm{id})(0)f'(0)$, $(Tf)'(0)=f(0)\lim_{n\to\infty}\mu_n^{\mathrm{id},f}$. This implies that $(Tf)'(0)=\alpha_1f'(0)=\alpha_2f(0)$, for every $f\in C^2[0,1]$ with $\alpha_1,\alpha_2\in\mathbb{T}$. Taking $f=\mathrm{id}$ gives $\alpha_1=0$, which leads to a contradiction.
(3) $Tf(0)=f'(0)\lim_{n\to\infty}\lambda_n^{1,f}$, $(Tf)'(0)=(T1)'(0)f(0)$, and $Tf(0)=f(0)\lim_{n\to\infty}\lambda_n^{\mathrm{id},f}$, $(Tf)'(0)=(T\mathrm{id})'(0)f'(0)$. In this case, we similarly arrive at a contradiction.
(4) $Tf(0)=f'(0)\lim_{n\to\infty}\lambda_n^{1,f}$, $(Tf)'(0)=(T1)'(0)f(0)$ and $Tf(0)=(T\mathrm{id})(0)f'(0)$, $(Tf)'(0)=f(0)\lim_{n\to\infty}\mu_n^{\mathrm{id},f}$.

For case (1), for every $f, g \in C^{(2)}[0,1]$ and $\alpha \in \mathbb{C}$, we have

$$\begin{aligned}
T(\alpha f+g)(x) &= (T(\alpha f+g))(0) + (T(\alpha f+g))'(0) + (S^2(D^2(T(\alpha f+g))(x) \\
&= (T1)(0)(\alpha f+g)(0) + (T\mathrm{id})'(0)(\alpha f+g)'(0) + (S^2(D^2(T(\alpha f+g))(x) \\
&= \alpha(T1)(0)f(0) + (T1)(0)g(0) + \alpha(T\mathrm{id})'(0)f'(0) + (T\mathrm{id})'(0)g'(0) \\
&+ \alpha(S^2(D^2(Tf)))(x) + (S^2(D^2(Tg)))(x) \\
&= \alpha Tf(0) + Tg(0) + \alpha(Tf)'(0) + (Tg)'(0) + \alpha(S^2(D^2(Tf)))(x) + (S^2(D^2(Tg)))(x) \\
&= \alpha Tf(x) + Tg(x).
\end{aligned}$$

Therefore, T is a linear map on $C^{(2)}[0,1]$. Similarly, we can prove that T is linear in case (4). Hence, T approximate local isometry. □

Theorem 4.2. *Every 2-local isometry of $C^{(2)}[0,1]$ is a local isometry.*

Proof. The proof follows in a similar manner to that of Theorem 4.1. □

By applying Theorem 4.2, we obtain an alternative proof of [14, Theorem 2.1], which asserts that any 2-local isometry on $C^{(2)}[0,1]$ must be a surjective linear isometry of $C^{(2)}[0,1]$.

Corollary 4.3. *Every 2-local isometry of $C^{(2)}[0,1]$ is a surjective linear isometry.*

Proof. Let T be 2-local isometry of $C^{(2)}[0,1]$. An application of Theorem 4.2 and Corollary 3.2 implies that T is surjective linear isometry of $C^{(2)}[0,1]$. □

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