Indian Institute of Information Technology Allahabad
Probability and Statistics (PAS)
C3 Review Test Tentative Marking Scheme

Program: B.Tech. 3^{rd} Semester (IT+ECE)
Duration: **02 Hours** — Full Marks: 46
Date: November 24, 2020 — Time:: 11:00 IST - 13:00 IST

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1. Let $F(\cdot)$ be any c.d.f. and $\lambda \in \mathbb{R}$. Find the set A such that $G(x) = (1+\lambda)F(x) - \lambda F^2(x)$ becomes a c.d.f. of some random variable for all $\lambda \in A$. [3]

   Solution. For all values of λ, $G(\cdot)$ is right continuous at every point, $\lim_{x\to\infty} G(x) = 1$, and $\lim_{x\to-\infty} G(x) = 0$. [1]

   For G to be a c.d.f., for $x_1 < x_2$,

   $$G(x_2) - G(x_1) = (F(x_2) - F(x_1))\,(1 + \lambda - \lambda(F(x_1) + F(x_2))) \geq 0, \qquad [1]$$

   which implies that

   $$1 + \lambda - \lambda(F(x_1) + F(x_2)) \geq 0, \ \because \ (F(x_2) - F(x_1)) \geq 0. \qquad [1]$$

   Case I: For $0 \leq \lambda \leq 1$, $1 + \lambda - \lambda(F(x_1) + F(x_2)) \geq 1 - \lambda > 0$. Hence G will be a c.d.f..

   Case II: For $-1 \leq \lambda < 0$, $1 + \lambda - \lambda(F(x_1) + F(x_2)) \geq 1 + \lambda \geq 0$. Hence G will be a c.d.f..

   Case III: For $\lambda > 1$, take an $\epsilon > 0$ with $\epsilon \leq \frac{\lambda-1}{\lambda}$. Now, we can choose x_1, x_2 and a c.d.f. F (e.g. uniform) such that $F(x_1) = 1 - \epsilon$, $F(x_2) = 1$.

   This implies that $1 + \lambda - \lambda(F(x_1) + F(x_2)) \leq 0$. So, G is not a c.d.f..

   Case IV: For $\lambda < -1$, take an $\epsilon > 0$ with $\epsilon \leq \frac{\lambda+1}{\lambda}$. Now, we can choose x_1, x_2 and a c.d.f. F (e.g. uniform) such that $F(x_1) = 0$, $F(x_2) = \epsilon$.

   This implies that $1 + \lambda - \lambda(F(x_1) + F(x_2)) \leq 0$. So, G is not a c.d.f..

   Combining all cases, we conclude that G is a c.d.f. for all $\lambda \in [-1, 1]$.

2. Suppose X follows Poisson distribution with parameter λ. Find the point x where $P(X = x)$ is maximum. [4]

   Solution. $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$ and $P(X = k-1) = \frac{e^{-\lambda}\lambda^{(k-1)}}{(k-1)!}$ [1]

   $\frac{P(X=k)}{P(X=k-1)} = \frac{\lambda}{k}$ [1]

   Thus, $P(X = x)$ is increasing when $x \leq \lambda$ and decreasing when $x \geq \lambda$. [1]

   Therefore, at $x = \lfloor \lambda \rfloor$, $P(X = x)$ is maximum. [1]

3. Let X has an exponential distribution with parameter θ. Let $Y = X - a$, where $a \geq 0$. Find a relation between $P(Y \leq x \mid X \geq a)$ and $P(X \leq x)$ for all x. [6]

   Solution.

   $$\begin{aligned} P(\{Y \leq x\} \cap \{X \geq a\}) &= P(\{X - a \leq x\} \cap \{X \geq a\}) \\ &= P(\{X \leq a + x\} \cap \{X \geq a\}). \end{aligned}$$

If $x < 0$, $P(\{X \leq a+x\} \cap \{X \geq a\}) = 0$ [1]

If $x \geq 0$, $P(\{X \leq a+x\} \cap \{X \geq a\}) = P(a \leq X \leq a+x) = e^{-a\theta}(1-e^{-\theta x})$. [1]

$P(X \geq a) = e^{-a\theta}$ [1]

$P(Y \leq x | X \geq a) = 1 - e^{-\theta x}$ [1]

$P(X \leq x) = 1 - e^{-\theta x}$, if $x \geq 0$ and $P(X < x) = 0$, if $x < 0$. [1]

Therefore, $P(Y \leq x \mid X \geq a) = P(X \leq x)$. [1]

4. Let Z be the standard normal random variable. Show that

$$P(|Z| \geq t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}, \ \forall \ t > 0. \quad [6]$$

Solution. Given that Z is a standard normal random variable, therefore

$$P(Z \geq t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{\frac{-x^2}{2}} dx. \quad [1]$$

Since $x/t > 1$ for $x > t$, we have [1]

$$P(Z \geq t) \leq \int_t^{\infty} \frac{x}{t} e^{\frac{-x^2}{2}} dx. \quad [1]$$

This gives

$$P(Z \geq t) \leq \frac{1}{\sqrt{2\pi}} \frac{e^{\frac{-t^2}{2}}}{t}. \quad [1]$$

Since Z is a symmetric random variable, we have $P(|Z| \geq t) = 2P(Z \geq t)$. [1]

Hence,

$$P(|Z| \geq t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}. \quad [1]$$

5. Let X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 24xy, & x > 0, y > 0, x+y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find $Var[X|Y=y]$. [5]

Solution: The marginal p.d.f. of Y is

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{1-y} 24xy\,dx = \begin{cases} 12y(1-y)^2, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$ [1]

The conditional p.d.f. of X, given $Y = y$ is

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{2x}{(1-y)^2}, & 0 < x < 1-y \\ 0, & \text{otherwise.} \end{cases}$ [1]

$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx = \int_0^{1-y} \frac{2x^2}{(1-y)^2} = \frac{2(1-y)}{3}$. [1]

$E[X^2|Y=y] = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)\,dx = \int_0^{1-y} \frac{2x^3}{(1-y)^2} = \frac{(1-y)^2}{2}$. [1]

$Var[X|Y=y] = E[X^2|Y=y] - (E[X|Y=y])^2 = \frac{(1-y)^2}{2} - \frac{4(1-y)^2}{9} = \frac{(1-y)^2}{18}$. [1]

6. Using Central Limit Theorem, find

$$\lim_{n\to\infty} e^{-nt}\sum_{k=1}^{n-1}\frac{(nt)^k}{k!},\ \forall\ t>0. \qquad [11]$$

Solution. Consider a sequence of $i.i.d$ random variables $\{X_n\}$ such that $X_n \sim Poisson(t)$ for all n. [1]

Then, $E(X_n) = Var(X_n) = t$ and $\sum_{i=1}^{n} X_i \sim Poisson(nt)$ for all n. $[\frac{1}{2}+\frac{1}{2}+1]$

Now,

$$\begin{aligned}
\lim_{n\to\infty} e^{-nt}\sum_{k=1}^{n-1}\frac{(nt)^k}{k!} &= \lim_{n\to\infty} P\left(1 \le \sum_{i=1}^{n} X_i \le (n-1)\right) && [1]\\
&= \lim_{n\to\infty} P\left(\frac{1-nt}{\sqrt{nt}} \le \frac{\sum_{i=1}^{n} X_i - nt}{\sqrt{nt}} \le \frac{(n-1)-nt}{\sqrt{nt}}\right) && [1]\\
&= \begin{cases} P(-\infty \le Z \le \infty), \text{ if } 0<t<1 \\ P(-\infty \le Z \le 0), \text{ if } t=1 \\ P(-\infty \le Z \le -\infty), \text{ if } t>1 \end{cases} && [1+1+1]\\
&= \begin{cases} 1, \text{ if } 0<t<1 \\ \frac{1}{2}, \text{ if } t=1 \\ 0, \text{ if } t>1 \end{cases} && [1+1+1]
\end{aligned}$$

7. One observation X is taken from a $N(0,\sigma^2)$ population. [4]

   (a) Find an unbiased estimator of σ^2.
   Solution. Given that X follows $N(0,\sigma^2)$.
   This implies $E(X) = 0$ and $Var(x) = E(X^2) - (E(X))^2$.
   Therefore, $E(X^2) = \sigma^2$. [1]
   Hence, X^2 is an unbiased estimator of σ^2. [1]

   (b) Find the method of moments estimator for σ.
   Solution. To obtain moment estimator, we have $E(X) = 0$,
   Equating the second population moment to the second sample moment we get,
   $E(X^2) = \sigma^2 = \sum_{i=1}^{n} X_i^2 = X^2$. [1]
   $\Rightarrow \sigma^2 = X^2 \Rightarrow \hat{\sigma} = |X|$.
   Hence, the moment estimator of σ is $|X|$. [1]

8. Let $X_1,\ldots,X_n$ be a random sample from the probability density function

$$f(x|\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\{-\lambda(x-\mu)^2/(2\mu^2 x)\},\ x>0.$$

Find the maximum likelihood estimator for μ and λ. [7]

Solution. Given that $X_1,\ldots,X_n$ is a random sample from the probability density function

$$f(x|\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\{-\lambda(x-\mu)^2/(2\mu^2 x)\},\ x>0.$$

The likelihood function is given by

$$L(\lambda, \mu|x) = \prod_{i=1}^{n} \left[\left(\frac{\lambda}{2\pi x_i^3} \right)^{1/2} \exp\left\{ -\lambda (x_i - \mu)^2 / (2\mu^2 x_i) \right\} \right]. \qquad [1]$$

Therefore, the log likelihood function is given by

$$\log L(\lambda, \mu|x) = l = \frac{n}{2} \log \lambda - \frac{n}{2} \log 2\pi - \frac{3}{2} \sum_{i=1}^{n} \log x_i - \frac{\lambda}{2\mu^2} \sum_{i=1}^{n} \left(\frac{(x_i - \mu)^2}{x_i} \right). \qquad [1]$$

The likelihood equation with respect to μ is given by $\frac{\partial l}{\partial \mu} = 0$.
This gives

$$\frac{-\lambda}{2} \frac{\partial}{\partial \mu} \left(\frac{1}{\mu^2} \sum_{i=1}^{n} \left(\frac{(x_i - \mu)^2}{x_i} \right) \right) = 0.$$

$\implies \hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}$. [2]
The likelihood equation with respect to λ is given by $\frac{\partial l}{\partial \lambda} = 0$.
This gives

$$\frac{n}{2\lambda} - \frac{1}{2\mu^2} \sum_{i=1}^{n} \left(\frac{(x_i - \mu)^2}{x_i} \right) = 0.$$

$\implies \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \left(\frac{1}{x_i} - \frac{1}{\bar{x}} \right)}$. [2]
Hence, the maximum likelihood estimator for μ and λ are $\bar{X}$ and $\frac{n}{\sum_{i=1}^{n} \left(\frac{1}{X_i} - \frac{1}{\bar{X}} \right)}$ respectively. [1]