INDIAN INSTITUTE OF INFORMATION TECHNOLOGY, ALLAHABAD

Mid Semester Examination: February 27, 2018

Paper code : SGNR210E
Paper Title: General Relativity
Paper Setter : Dr. Srijit Bhattacharjee

Max Marks : 30 **Duration : 2 hours**

Attempt all the parts of a question at the same place. Number indicated on the right in [ ] is full marks of that particular problem. There is **no credit** for an answer if proper justification is not given, even if the answer is correct. Notations are standard. **Do not write anything** on question paper and cover pages except your details.

1. (a) Show that the closed interval $A = [a, b]$ is homeomorphic to the closed unit interval $I = [0, 1]$.
(b) Prove that image of a compact topological space X under a continuous map $f : X \to Y$, is also compact. [2+3]

2. Show that unit 2-sphere(S^2) is a differentiable manifold by constructing a 2-charted atlas. Find the normalized basis vectors of the tangent space at any point p on S^2. [6+4]

3. (a) Find the components of the commutator of two smooth vector fields U and V in a manifold, in any coordinate basis.
(b) Let U, V, W be three smooth vector fields on a manifold. Show that their commutators satisfy the Jacobi identity:
$$[[U, V], W] \; + \; [[V, W], U] \; + \; [[W, U], V] = 0.$$
(c) Let $Y_1, ..., Y_n$ be smooth vector fields on an n-dimensional manifold M such that at each $p \in M$ they form a basis of the tangent space $T_pM \equiv V_p$. Then, at each point, we may expand each commutator $[Y_\alpha, Y_\beta]$ in this basis, thereby defining the functions $C^\gamma_{\alpha\beta} = -C^\gamma_{\beta\alpha}$ by
$$[Y_\alpha, Y_\beta] = C^\gamma_{\alpha\beta} Y_\gamma.$$
Use Jacobi identity to derive an equation satisfied by the functions $C^\gamma_{\alpha\beta}$.
(d) Let $v_1.....v_n$ be a basis of the vector space V, and let $v^{1*}, ...v^{n*}$ be its dual basis. Let $X \in V$ and let $\omega \in V^*$. Show that
$$\begin{aligned} X &= \Sigma_\alpha v^{\alpha *}(X) v_\alpha, \\ \omega &= \Sigma_\alpha \omega(v_\alpha) v^{\alpha *}. \end{aligned}$$
[2+3+2+3]

4. Suppose f be a differentiable function on $\Re^3$ defined as,
$$f(x, y, z) = (x^2 - 1)y + (y^2 + 2)z.$$
Let $V_p = (v_1, v_2, v_3)$ be a smooth vector field on $\Re^3$. Find $V_p[f]$, at $\mathbf{p} = (\mathbf{1}, -\mathbf{1}, \mathbf{2})$. Evaluate the same for $V_p = (1, 2, -3)$. What is the interpretation of df in a differentiable manifold? [5]