General Relativity
Problem Set II
Differentiable forms, Push-forward, Pull-back

1. The map $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by

$$h(a, b) := (a^2 - 2b, 4a^3 b^2)$$

Compute the push-forward of the derivation $h_*(4x^1 \frac{\partial}{\partial x^1} + 3x^2 \frac{\partial}{\partial x^2})_{(1,2)}$ at the point $(1, 2) \in \mathbb{R}^2$, where $\{x^1, x^2\}$ are the natural coordinates on $\mathbb{R}^2$.

2. Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be rotation counterclockwise by an angle θ. Let ∂_x, ∂_y be the coordinate vector fields on $\mathbb{R}^2$. Show that at any point of $\mathbb{R}^2$

$$\begin{aligned} h_* \partial_x &= \cos\theta \partial_x - \sin\theta \partial_y \\ h_* \partial_y &= \sin\theta \partial_x + \cos\theta \partial_y \end{aligned}$$

3. Prove that exterior derivative d and pull-back f^* commutes. First show for a function and then for a r-form.

4. Let, $f_1 = x^2 - y^2$, and $f_2 = 2xy$. Also, $\omega_1 = xydx + xdy, \omega_2 = -2ydx + xdy, \omega_3 = (x^2 + y^2)dy$. Let, $f : \mathbb{R} \rightarrow \mathbb{R}^2$ s.t. $(x, y) = f(t) = (t, t^2)$. Find

   (a) $f^*(f_1)$, (b)$f^*(f_2)$, (c) $f^*(\omega_1)$, (d) $f^*(\omega_2)$, (e) $f^*(\omega_3)$.

   Show $f^*(df_{1(2)}) = df^*(f_{1(2)})$ and $f^*(d\omega_1) = d(f^*\omega_1)$.

5. Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be rotation counterclockwise by an angle θ. Let dx, dy be the usual basis of 1-forms in $\mathbb{R}^2$. Show that at any point of $\mathbb{R}^2$

$$\begin{aligned} h^* dx &= \cos\theta dx - \sin\theta dy \\ h^* dy &= \sin\theta dx + \cos\theta dy \end{aligned}$$

6. Let $f : \mathbb{R} \rightarrow \mathbb{R}^3$, and $f(t) = (\sin t, \cos t, t)$. Let, ω is a one-form

$$\omega(x, y, z) = 3(x^2 + y^2)dx - 2xdy + z^2 dz.$$

Find out $f^*\omega(t)$.