**General Relativity**
**Problem Set I**
**Topological Spaces and Differentiable Manifolds**

1. Let, $X = \{a, b, c, d, e, f\}$. Let $\mathcal{C} = \{\phi, X, \{a, b\}, \{b\}, \{b, c\}, \{a, b, c\}\}$. Show that this collection defines a topology on $X$. If we leave out $\{a, b\}$ in $\mathcal{C}$, do we still get a topology? What if we leave out $\{b\}$? What if we add $\{d\}$?

2. Does $d(x, y) = (x - y)^2$ define a metric on $R$?

3. List all the topologies on $X = \{a, b, c\}$, with consist of exactly four members. (Hint: consider two cases. i) two open sets which are partition of $X$, ii) when those are not paritions of $X$.)

4. Consider the function $f : R \to R$. defined by

$$f(x) = \begin{cases} x - 1 & \text{if } x \leq 3 \\ \frac{1}{2}(x + 5) & x > 3 \end{cases}$$

   Show that $f$ is not continuous.

5. Prove that a function $f : X \to Y$ is continuous if and only if the inverse image of every closed subset of Y is a closed subset of $X$.

6. Show that us $T^2 = S^1 \times S^1$ is a differentiable manifold.

7. Show that $S^n$ is a Differentiable manifold.

8. Argue that two diffeomorphic spaces/manifolds must be of same dimension.

9. Let $M$ be the surface $y_3 = y_1^2 + y_2^2$ , which we paramaterize by

$$\begin{aligned} y_1 &= x^1, \\ y_2 &= x^2, \\ y_3 &= (x^1)^2 + (x^2)^2 \end{aligned}$$

   Specify a curve on this surface and hence find out a tangent vector field on the manifold.