Indian Institute of Information Technology Allahabad
Probability and Statistics (SPAS230C)
Back Paper Examination - 2016 : Marking Scheme

Numbers indicated on the right in red [ ] are marks that may be awarded if that particular step is done correctly.

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1. Provide a short proof or answer of the following.

   (a) Let A, $B \in \mathcal{F}$ and $P(A) = 0$. Then $P(A^c \cup B) = 1$. [2]
   Soln: $A^c \subseteq A^c \cup B \implies P(A^c) \leq P(A^c \cup B)$ and $P(A) = 0 \implies P(A^c) = 1$. [1]
   Thus $1 = P(A^c) \leq P(A^c \cup B) \leq 1 \implies P(A^c \cup B) = 1$ [1]

   (b) Let A and B be two events such that $P(A) = p_1 > 0$, $P(B) = p_2 > 0$ and $p_1 + p_2 > 1$. Show that $P(B|A) \geq 1 - \dfrac{1-p_2}{p_1}$. [2]
   Soln: $P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(A)+P(B)-P(A\cup B)}{P(A)}$ [1]
   $\geq \frac{P(A)+P(B)-1}{P(A)}$ (since $P(A \cup B) \leq 1$). [1]
   Thus $P(B|A) \geq \frac{p_1+p_2-1}{p_1} = 1 - \frac{1-p_2}{p_1}$.

   (c) Let X be a random variable such that $E(X) = 3$ and $E(X^2) = 13$, then determine a lower bound for $P(-2 < X < 8)$. [4]
   Soln: $\sigma^2 = Var(X) = E(X^2) - (E(X))^2 = 13 - 9 = 4$. [1]
   $P(-2 < X < 8) = P(-5 < X - 3 < 5) = P(|X - 3| < 5) = 1 - P(|X - 3| \geq 5)$. [1]
   By Chebyshev Inequality $P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$ [1]
   we have $P(|X - 3| \geq 5) \leq \frac{4}{25}$
   $\implies P(-2 < X < 8) \geq 1 - \frac{4}{25} = \frac{21}{25}$
   Thus a lower bound is $\frac{21}{25}$. [1]

   (d) If $M_X(t) = e^{ct}$ for $t \in \mathbb{R}$, where c is a constant. Find the variance of X? [2]
   Soln: $E(X) = ce^{ct}|_{t=0} = c$, $E(X^2) = c^2e^{ct}|_{t=0} = c^2$. [1]
   Hence, $Var(X) = E(X^2) - (E(X))^2 = 0$. [1]

2. Let $\Omega = \{0, 1, 2, \ldots\}$ and $\mathcal{F} = \mathcal{P}(\Omega)$, the power set of Ω. Define $P : \mathcal{F} \to \mathbb{R}$ by

$$P(A) = \sum_{x\in A} p(1-p)^x, \text{ for } 0 < p < 1.$$

   Prove that P is a probability function on $(\Omega, \mathcal{F})$. [4]
   Soln:

   (a) For $A \in \mathcal{F}$, $P(A) = \sum_{x\in A} p(1-p)^x \geq 0$ (since $p > 0$, and $(1-p) > 0$). [1]

   (b) $P(\Omega) = \sum_{x\in\Omega} p(1-p)^x = p\sum_{x=0}^{\infty}(1-p)^x = \frac{p}{1-(1-p)} = 1$. [1]

(c) Let $A_1, A_2, \ldots$ be a countably infinite collection of mutually exclusive events. Then

$$\begin{aligned} P\left(\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{x \in \bigcup\limits_{i=1}^{\infty} A_i} p(1-p)^x \\ &= \sum_{i=1}^{\infty} \sum_{x \in A_i} p(1-p)^x && [1] \\ &= \sum_{i=1}^{\infty} P(A_i). && [1] \end{aligned}$$

Therefore, P is a probability function on $(\Omega, \mathcal{F})$.

3. L Let X be a random variable having a binomial distribution with probability of success $p \in (0,\, 1)$. Find the moment generating function of X. Mention its' domain explicitly [3]

Soln: The moment generating function of X is given by

$$\begin{aligned} M_X(t) &= E(e^{tX}) && [1] \\ &= \sum_{x=0}^{n} e^{tx} f_X(x) \\ &= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} && [1] \\ &= (pe^t + q)^n \qquad \forall t \in \mathbb{R} && [1] \end{aligned}$$

4. Let X be a discrete random variable with probability mass function

$$f_X(x) = \begin{cases} \frac{1}{3}(\frac{2}{3})^x & \text{if } x \in \{0, 1, 2, \ldots\} \\ 0 & \text{otherwise.} \end{cases}$$

Then show that $Y = \dfrac{X}{X+1}$ is a discrete random variable and hence find the probability mass function of Y. [5]

Soln: Let $h : \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R}$ be a function defined by $h(x) = \frac{x}{x+1}$. Then clearly h is continuous function and hence a Borel function. So $Y = h(X)$ is a discrete random variable with support $S_Y = \{y = \frac{i}{i+1}, i \in \{0, 1, \ldots\}\}$. [1+1]

Therefore, for $y \notin S_Y$, the probability mass function of Y is $f_Y(y) = 0$. [1]

For $y \in S_Y$, the probability mass function of Y,

$$
\begin{aligned}
f_Y(y) &= P\left(\left\{\frac{X}{X+1} = y\right\}\right) \\
&= P\left(\{X = y(X+1)\}\right) \\
&= P\left(\left\{X = \frac{y}{1-y}\right\}\right) && [1] \\
&= \begin{cases} \frac{1}{3}(\frac{2}{3})^i & \text{if } y \in S_Y \\ 0 & \text{otherwise.} \end{cases} && [1]
\end{aligned}
$$

Thus the probability mass function of Y is

$$
f_Y(y) = \begin{cases} \frac{1}{3}(\frac{2}{3})^i & \text{if } y \in S_Y \\ 0 & \text{otherwise.} \end{cases}
$$

5. Let X be a random variable with probability density function

$$
f_X(x) = \begin{cases} \theta e^{-\theta x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}
$$

where $\theta > 0$. Find the probability density function of $Y = (X - \frac{1}{\theta})^2$. [7]

Soln: Define a function $h(x) = (x - \frac{1}{\theta})^2$ then note that $h(x)$ is a continuous function. This implies $Y = (X - \frac{1}{\theta})^2$ is a random variable.
Consider $S_1 = (0, \frac{1}{\theta})$ and $S_2 = (\frac{1}{\theta}, \infty)$ so that $S_X = (0, \infty) = S_1 \cup S_2$. [1]
Now, $h(x) = (x - \frac{1}{\theta})^2$ is strictly decreasing in S_1 with inverse $h_1^{-1}(y) = \frac{1}{\theta} - \sqrt{y}$. [1]
Now, $h(x) = (x - \frac{1}{\theta})^2$ is strictly increasing in S_2 with inverse $h_2^{-1}(y) = \frac{1}{\theta} + \sqrt{y}$. [1]

Then the probability density function of Y is given by

$$
\begin{aligned}
f_Y(y) &= \begin{cases} f_X(h_1^{-1}(y))|\frac{d}{dy}(h_1^{-1}(y))| + f_X(h_2^{-1}(y))|\frac{d}{dy}(h_2^{-1}(y)) & \text{if } 0 < y < \frac{1}{\theta^2} \quad [1] \\ f_X(h_2^{-1}(y))|\frac{d}{dy}(h_2^{-1}(y))|, & \text{if } y > \frac{1}{\theta^2} \quad [1] \\ 0 & \text{otherwise} \quad [1] \end{cases} \\
&= \begin{cases} \frac{\theta}{2e\sqrt{y}}(e^{\theta\sqrt{y}} + e^{-\theta\sqrt{y}}) & \text{if } 0 < y < \frac{1}{\theta^2} \\ \frac{\theta}{2e\sqrt{y}}e^{-\theta\sqrt{y}} & \text{if } y > \frac{1}{\theta^2} \\ 0 & \text{otherwise.} \end{cases} && [1]
\end{aligned}
$$

6. Let (X, Y) be a random vector with joint probability density function

$$
f(x, y) = \begin{cases} x^2 + \frac{xy}{3}, & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Find all moments of order 2. Also find the correlation coefficient between X and Y. [7]

Soln: $E(X^2) = \int_0^1 \int_0^1 x^2 f(x, y) dx dy = \frac{29}{120}$. [1]

$E(XY) = \int_0^1 \int_0^1 xyf(x,y)dxdy = \frac{35}{216}$. [1]

$E(Y^2) = \int_0^1 \int_0^1 y^2 f(x,y)dxdy = \frac{11}{72}$. [1]

$E(X) = \int_0^1 \int_0^1 xf(x,y)dxdy = \frac{11}{36}$. [1]

$E(Y) = \int_0^1 \int_0^1 yf(x,y)dxdy = \frac{2}{9}$. [1]

$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{61}{648}$. [1]

Now $\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} == \dfrac{61\sqrt{10}}{\sqrt{961}\sqrt{67}}$. [1]

7. Let $\underline{X} = (X_1, X_2)$ be a random vector with joint probability density function

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} \frac{1+x_1x_2}{4}, & |x_1| < 1, |x_2| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X_1 and X_2 independent? [5]

Soln: The marginal probability density function of X_1 is given by

$$\begin{aligned} f_{X_1}(x_1) &= \int_{-\infty}^{\infty} f_{\underline{X}}(x_1,\, x_2)dx_2 \\ &= \begin{cases} \displaystyle\int_{-1}^{1} \frac{1+x_1x_2}{4} dx_2 & \text{if } |x_1| < 1 \\ 0 & \text{otherwise} \end{cases} \qquad [1] \\ &= \begin{cases} \frac{1}{2} & \text{if } |x_1| < 1 \\ 0 & \text{otherwise.} \end{cases} \qquad [1] \end{aligned}$$

The marginal probability density function of X_2 is

$$\begin{aligned} f_{X_2}(x_2) &= \int_{-\infty}^{\infty} f_{\underline{X}}(x_1,\, x_2)dx_1 \\ &= \begin{cases} \displaystyle\int_{-1}^{1} \frac{1+x_1x_2}{4} dx_2 & \text{if } |x_2| < 1 \\ 0 & \text{otherwise} \end{cases} \qquad [1] \\ &= \begin{cases} \frac{1}{2} & \text{if } |x_2| < 1 \\ 0 & \text{otherwise.} \end{cases} \qquad [1] \end{aligned}$$

Clearly $f_{\underline{X}}(x_1, x_2) \neq f_{X_1}(x_1) f_{X_2}(x_2)$. Hence X_1 and X_2 are not independent. [1]

8. Let $\underline{X} = (X_1,\, X_2)$ be a random vector with probability density function

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} \frac{1}{2}e^{-x_1} & \text{if } 0 < x_2 < x_1 < \infty \\ \frac{1}{2}e^{-x_2} & \text{if } 0 < x_1 < x_2 < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find the joint probability density function of $Y_1 = X_1 + X_2$ and $Y_2 = \dfrac{X_2}{X_1 + X_2}$ using transformation technique. [7]

Soln: Let $g(x_1,\, x_2) = (x_1 + x_2,\, \dfrac{x_2}{x_1 + x_2})$ be a one-to-one function from the range of random vector $\underline{X} = (X_1,\, X_2)$ to $\mathbb{R}^2$ with inverse function $h(y_1,\, y_2) = (h_1(y_1,\, y_2),\, h_2(y_1,\, y_2)) = (y_1(1 - y_2),\, y_1 y_2)$. [2]
The Jacobian of the transformation is

$$J = \begin{vmatrix} \dfrac{\partial h_1(y_1,\, y_2)}{\partial y_1} & \dfrac{\partial h_1(y_1,\, y_2)}{\partial y_2} \\ \dfrac{\partial h_2(y_1,\, y_2)}{\partial y_1} & \dfrac{\partial h_2(y_1,\, y_2)}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 - y_2 & -y_1 \\ y_2 & y_1 \end{vmatrix} = y_1 \qquad [1]$$

Since $J = y_1$ is different from zero in range of transformation, and g and h are continuous, $(Y_1,\, Y_2)$ is an absolutely continuous random vector with probability density function [1]

$$\begin{aligned} f_{(Y_1,Y_2)}(y_1, y_2) &= f_{(X_1,\,X_2)}\left(h_1(y_1,\, y_2),\, h_2(y_1,\, y_2)\right)|J| \\ &= |y_1| f_{(X_1,\,X_2)}\left(y_1(1 - y_2),\, y_1 y_2\right) \qquad [1] \\ &= \begin{cases} \dfrac{1}{2}|y_1|e^{-y_1(1-y_2)} & \text{if } 0 < y_1 y_2 < y_1(1 - y_2) < \infty \\ \dfrac{1}{2}|y_1|e^{-y_1 y_2} & \text{if } 0 < y_1(1 - y_2) < y_1 y_2 < \infty \\ 0 & \text{otherwise} \end{cases} \qquad [1{+}1] \end{aligned}$$

9. Let $X_1,\, X_2,\, \cdots,\, X_k$ be k (fixed positive integer) absolutely continuous random variables with probability density functions $f_1(\cdot),\, f_2(\cdot),\, \ldots,\, f_k(\cdot)$. Let $c_i \geq 0,\, i = 1,\, 2,\, \ldots,\, k$, be real constant such that $\sum_{i=1}^{k} c_i = 1$.

   (a) Show that

   $$f(x) = \sum_{i=1}^{k} c_i f_i(x)$$

   is a probability density function of a random variable. [2]
   Soln: As $f_i(x) \geq 0,\, \forall x \in \mathbb{R},\, i = 1,\, 2,\, \ldots,\, k,\, c_i \geq 0,\, i = 1,\, 2,\, \ldots,\, k,$

   $$f(x) = \sum_{i=1}^{k} c_i f_i(x) \geq 0,\, \forall x \in \mathbb{R}. \qquad [1]$$

   Also $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty}\{\sum_{i=1}^{k} c_i f_i(x)\}dx = \sum_{i=1}^{k} c_i\{\int_{-\infty}^{\infty} f_i(x)dx\} = 1$ (as $\int_{-\infty}^{\infty} f_i(x)dx = 1,\, i = 1,\, 2,\, \ldots,\, k$, and $\sum_{i=1}^{k} c_i = 1$). [1]

   (b) Let X be the absolutely continuous random variable with probability density function $f(\cdot)$ as given in part (a). Show that

   $$\mu = \sum_{i=1}^{k} c_i \mu_i,$$

where $\mu = E(X)$ and $\mu_i = E(X_i)$, $i = 1, 2, \ldots, k$, provided all the expectations involved exists. [2]

Soln:

$$\begin{aligned}
\mu = E(X) &= \int_{-\infty}^{\infty} x f(x) dx \qquad [1] \\
&= \int_{-\infty}^{\infty} x(\sum_{i=1}^{k} c_i f_i(x)) dx \\
&= \sum_{i=1}^{k} c_i (\int_{-\infty}^{\infty} x f_i(x)) dx) \\
&= \sum_{i=1}^{k} \mu_i c_i \quad (as \quad \mu_i = \int_{-\infty}^{\infty} x f_i(x)) dx) \qquad [1]
\end{aligned}$$

10. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A, B \in \mathcal{F}$. Define X and Y so that

$$X(\omega) = I_A(\omega), \ Y(\omega) = I_B(\omega) \ \ \forall \ \omega \in \Omega.$$

(a) Show that (X, Y) is a discrete type random vector. [5]

Soln: The probability function of (X, Y) is

$$P(X = x, Y = y) = \begin{cases} P(A^c \cap B^c) & \text{if } x = 0, \ y = 0 \\ P(A \cap B^c) & \text{if } x = 1, \ y = 0 \\ P(A^c \cap B) & \text{if } x = 0, \ y = 1 \\ P(A \cap B) & \text{if } x = 1, \ y = 1 \\ 0 & \text{otherwise.} \end{cases} \qquad [4]$$

Since $S_{(X, Y)} = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ is countable and

$$\sum_{(x, y) \in S_{(X, Y)}} P(X = x, Y = y) = 1,$$

(X, Y) is a discrete type random vector. [1]

(b) Using part (a), show that X and Y are independent if and only if A and B are independent. [6]

Soln:$(\Rightarrow)$ Assume that X and Y are independent. So $P(X = x, Y = y) = P(X = x)P(Y = y)$. [1]

In particular, $P(X = 1, Y = 1) = P(X = 1)P(Y = 1)$. Hence from Part (a), $P(A \cap B) = P(A)P(B)$. Therefore A and B are independent. [1]

$(\Leftarrow)$ Assume that A and B are independent. So $P(A \cap B) = P(A)P(B)$, $P(A^c \cap B) = P(A^c)P(B)$, $P(A \cap B^c) = P(A)P(B^c)$, $P(A^c \cap B^c) = P(A^c)P(B^c)$. [3]

So from Part (a), it is clear that $P(X = x, Y = y) = P(X = x)P(Y = y)$ i. e. X and Y are independent. [1]

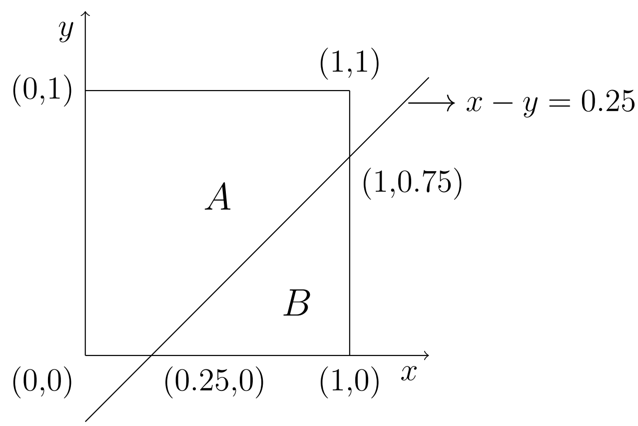
11. A bus and a passenger arrive on a bus-stop at uniformly distributed time over the time interval 0 to 1 hour. Assume that the arrival times of the bus and passenger are independent of one another. The passenger will wait up to 15 minutes for the bus to arrive. What is the probability that the passenger will take the bus? [7]

Soln: Let X and Y be the times of the arrival of bus and the passenger, respectively. We need to find

$$P\left(0 \le X - Y \le 0.25\right), \qquad [1]$$

where X and Y are independent. Also $X \sim \mathrm{U}(0,\, 1)$ and $Y \sim \mathrm{U}(0,\, 1)$. Hence joint PDF of $(X,\, Y)$ is given by

$$f_{(X,Y)}(x,\, y) = \begin{cases} 1 & \text{if } 0 < x < 1,\; 0 < y < 1 \qquad [1] \\ 0 & \text{otherwise.} \qquad [1] \end{cases}$$



Hence

$$\begin{aligned} P\left(0 \le X - Y \le 0.25\right) &= \int\int_A f_{(X,Y)}(x,\, y)\,dxdy && [1] \\ &= \int\int_A 1\,dxdy \\ &= \text{Area of } A && [1] \\ &= 1 - \text{Area of } B && [1] \\ &= 1 - 0.5 \times 0.75 \times 0.75 \\ &= 0.71875. && [1] \end{aligned}$$

12. You enter a special kind of chess tournament, in which you play one game with each of three opponents, but you get to choose the order in which you play your opponents, knowing the probability of a win against each. You win the tournament if you win two games in a row, and you want to maximize the probability of winning. Show that it is optimal to play the weakest opponent second, and that the order of playing the other two opponents does not matter? [10]

Soln: Let $s < m < w$ be the probability of winning a game by myself against other three players. For simplicity we are labelling the strongest player as player 1, the weekest player as player 3, and the the other player as player 2.

Consider the following table

Sl. No.	Order of the play	Probability of winning the tournament
1.	(1, 2, 3)	sm + (1-s)mw = sm + mw -smw [1]
2.	(3, 2, 1)	wm + (1-w)ms = sm + mw -smw [1]
3.	(2, 1, 3)	ms + (1-m)sw = ms + sw -smw [1]
4.	(3, 1, 2)	ws + (1-w)ms = ms + sw -smw [1]
5.	(1, 3, 2)	sw + (1-s)wm = sw + mw -smw [1]
6.	(2, 3, 1)	mw + (1-m)ws = sw + mw -smw [1]

It is clear from the table that probability of winning the tournament is depends only on the choice of the opponent in the second game. [2]

The choice of the opponents in first and third games are immaterial. Hence I need to compare the probabilities in Sl. No. 1, 3 and 5.

Now $s < w \Rightarrow ms < mw \Rightarrow ms + sw - smw < mw + sw - smw \Rightarrow$ Prob. in Sl. No. 3 $<$ Prob. in Sl. No. 5. [1]

Also $m < w \Rightarrow sm < sw \Rightarrow sm + mw - smw < sw + mw - smw \Rightarrow$ Prob. in Sl. No. 1 $<$ Prob. in Sl. No. 5.

Hence the optimal choice is to play the weakest opponent second. [1]