Indian Institute of Information Technology Allahabad
Probability and Statistics (SPAS230C)
Back Paper Examination - 2016

Duration: 3 Hours
Full Marks: 80
Instructor: ABAB/SU/AT

Date: July 18, 2016
Time: 14:00 - 17:00 IST
BTech - II Semester (IT+ECE+RGIT) and IV (IT+RGIT)

Attempt **all** the questions. Numbers indicated on the right in [ ] are full marks of that particular problem. There is **no credit** for a solution if the appropriate work is not shown, even if the answer is correct. Notations are standard. **Do not write** on question paper and cover pages except the your detail. This question paper has **three** pages.

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1. Provide a short proof or answer of the following.

   (a) Let A, $B \in \mathcal{F}$ and $P(A) = 0$. Then $P(A^c \cup B) = 1$. [2]

   (b) Let A and B be two events such that $P(A) = p_1 > 0$, $P(B) = p_2 > 0$ and $p_1 + p_2 > 1$. Show that $P(B|A) \geq 1 - \dfrac{1 - p_2}{p_1}$. [2]

   (c) Let X be a random variable such that $E(X) = 3$ and $E(X^2) = 13$, then determine a lower bound for $P(-2 < X < 8)$? [4]

   (d) If $M_X(t) = e^{ct}$ for $t \in \mathbb{R}$, where c is a constant. Find the variance of X? [2]

2. Let $\Omega = \{0, 1, 2, \ldots\}$ and $\mathcal{F} = \mathcal{P}(\Omega)$, the power set of Ω. Define $P : \mathcal{F} \to \mathbb{R}$ by

$$P(A) = \sum_{x \in A} p(1-p)^x, \text{ for } 0 < p < 1.$$

   Prove that P is a probability function on $(\Omega, \mathcal{F})$. [4]

3. Let X be a random variable having a binomial distribution with probability of success $p \in (0,\ 1)$. Find the moment generating function of X. Mention its' domain explicitly. [3]

4. Let X be a discrete random variable with probability mass function

$$f_X(x) = \begin{cases} \frac{1}{3}(\frac{2}{3})^x & \text{if } x \in \{0, 1, 2, \ldots\} \\ 0 & \text{otherwise.} \end{cases}$$

   Let $Y = \dfrac{X}{X+1}$ then show that Y is discrete random variable and hence find the probability mass function of Y. [5]

5. Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} \theta e^{-\theta x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

   where $\theta > 0$. Find the probability density function of $Y = (X - \frac{1}{\theta})^2$. [7]

6. Let (X,Y) be a random vector with joint probability density function

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3}, & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find all moments of order 2. Also find the correlation coefficient between X and Y. [7]

7. Let $\underline{X} = (X_1, X_2)$ be a random vector with joint probability density function

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} \frac{1+x_1x_2}{4}, & |x_1| < 1, |x_2| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X_1 and X_2 independent? [5]

8. Let $\underline{X} = (X_1,\, X_2)$ be a random vector with probability density function

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} \frac{1}{2}e^{-x_1} & \text{if } 0 < x_2 < x_1 < \infty \\ \frac{1}{2}e^{-x_2} & \text{if } 0 < x_1 < x_2 < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find the joint probability density function of $Y_1 = X_1 + X_2$ and $Y_2 = \dfrac{X_2}{X_1 + X_2}$ using transformation technique. [7]

9. Let $X_1,\, X_2,\, \cdots,\, X_k$ be k (fixed positive integer) absolutely continuous random variables with probability density functions $f_1(\cdot),\, f_2(\cdot),\, \ldots,\, f_k(\cdot)$. Let $c_i \geq 0$, $i = 1,\, 2,\, \ldots,\, k$, be real constant such that $\sum_{i=1}^{k} c_i = 1$.

   (a) Show that

   $$f(x) = \sum_{i=1}^{k} c_i f_i(x)$$

   is a probability density function of a random variable. [2]

   (b) Let X be the absolutely continuous random variable with probability density function $f(\cdot)$ as given in part (a). Show that

   $$\mu = \sum_{i=1}^{k} c_i \mu_i,$$

   where $\mu = E(X)$ and $\mu_i = E(X_i)$, $i = 1,\, 2,\, \ldots,\, k$, provided all the expectations involved exists. [2]

10. Let $(\Omega,\, \mathcal{F},\, P)$ be a probability space and $A,\, B \in \mathcal{F}$. Define X and Y so that

$$X(\omega) = I_A(\omega),\ Y(\omega) = I_B(\omega) \quad \forall\, \omega \in \Omega.$$

   (a) Show that $(X,\, Y)$ is a discrete type random vector. [5]

   (b) Using part (a), show that X and Y are independent if and only if A and B are independent. [6]

11. A bus and a passenger arrive on a bus-stop at uniformly distributed time over the time interval 0 to 1 hour. Assume that the arrival times of the bus and passenger are independent of one another. The passenger will wait up to 15 minutes for the bus to arrive. What is the probability that the passenger will take the bus? [7]

12. You enter a special kind of chess tournament, in which you play one game with each of three opponents, but you get to choose the order in which you play your opponents, knowing the probability of a win against each. You win the tournament if you win two games in a row, and you want to maximize the probability of winning. Show that it is optimal to play the weakest opponent second, and that the order of playing the other two opponents does not matter? [10]

Some Useful Results

The PMF/PDF are mentioned only for respective support

Distribution	PMF/PDF
Binomial(n, p)	$\binom{n}{x} p^x (1-p)^{n-x},\ x = 0, 1, \ldots, n$
Continuous Uniform (a, b)	$1/(b-a),\ x \in (a, b)$