Indian Institute of Information Technology Allahabad
Complex Analysis and Integral Transformation (SMAT330)

Back Paper Examination - Tentative Marking Scheme

Numbers indicated on the right in red [ ] are marks that may be awarded if that particular step is done correctly.

Notations:

1. $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $i^2 = -1$.
2. $\mathcal{F}(s) = \mathcal{L}(f(t))$ denotes the Laplace transform of a function f and $\mathcal{L}^{-1}(\mathcal{F}(s)) = f(t)$ denote the inverse Laplace transform.
3. $\mathfrak{F}_{\mathfrak{s}}\{f(x)\}$ and $\mathfrak{F}_{\mathfrak{c}}\{f(x)\}$ denotes the Fourier sine and cosine transforms respectively.

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1. Provide a short proof or answer of the following statements.

   (a) The set $\{z \in \mathbb{C} : \frac{1}{2} < |z| < \frac{7}{3}\}$ is connected. [1]
   Solution. The set $S = \{z \in \mathbb{C} : \frac{1}{2} < |z| < \frac{7}{3}\}$ is connected since each pair of points z_1 and z_2 in it can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in S. [1]

   (b) Suppose f and g are piecewise continuous functions on $[0, \infty]$, and have exponential order α and β respectively. Then $\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$, where a and b are arbitrary constants. [2]
   Solution. Since $\mathcal{L}(f)$ exists for $\mathcal{R}e(s) > \alpha$ and $\mathcal{L}(g)$ exists for $\mathcal{R}e(s) > \beta$, $\mathcal{L}(f+g)$ exists for $\mathcal{R}e(s) > \max\{\alpha, \beta\}$. Moreover, [1]
   $\mathcal{L}(af+bg) = \int_0^\infty e^{-st}(af(t)+bg(t))dt = a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt = a\mathcal{L}(f) + b\mathcal{L}(g)$. [1]

   (c) Let $f(x)$ be continuous and absolutely integrable on the x-axis, $f'(x)$ piecewise continuous on every finite interval, and $\lim_{x\to\infty} f(x) = 0$. Then
   $$\mathfrak{F}_{\mathfrak{s}}\{f'(x)\} = -w\mathfrak{F}_{\mathfrak{c}}\{f(x)\}. \qquad [3]$$
   Solution.
   $$\begin{aligned} \mathfrak{F}_{\mathfrak{s}}\{f'(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x)\sin wx\, dx & [1] \\ &= \sqrt{\frac{2}{\pi}} \left[f(x)\sin wx \Big|_0^\infty - w\int_0^\infty f(x)\cos wx\, dx \right] & [1] \\ &= -w\mathfrak{F}_{\mathfrak{c}}\{f(x)\}. & [1] \end{aligned}$$

   (d) Any bounded function has exponential order. [1]
   Solution. Any bounded function f has exponential order 0 as there exists a constant M such $|f(t)| \leq M, \ \forall\, t$. [1]

(e) Evaluate $\displaystyle\int_{|z-i|=2} \frac{e^z}{z(z+3)} dz.$ [4]

Solution. $\int_{|z-i|=2} \frac{e^z}{z(z+3)} dz = \frac{1}{3}\int_{|z-i|=2} \frac{e^z}{z} dz - \frac{1}{3}\int_{|z-i|=2} \frac{e^z}{z+3} dz$ [1]

By Cauchy integral formula, $\int_{|z-i|=2} \frac{e^z}{z} dz = 2\pi i$ [1]

and by Cauchy's Theorem, $\int_{|z-i|=2} \frac{e^z}{z+3} dz = 0.$ [1]

Hence, $\int_{|z-i|=2} \frac{e^z}{z(z+3)} dz = \frac{2\pi i}{3}.$ [1]

(f) Does there exists a function f such that $\mathcal{L}(f(t)) = \dfrac{s}{\log s}$. [2]

Solution. No. If $\mathcal{F}(s) = \mathcal{L}(f(t))$, then $\lim\limits_{s\to\infty} \mathcal{F}(s) = 0.$ [2]

(g) Lef f be an entire function such that $f(z) > M$, $\forall z \in \mathbb{C}$, for some constant M. Then f is a constant function. [4]

Solution. Since $|f(z)| > 0$, $\forall\ z \in \mathbb{C}$, $f(z) \neq 0$, $\forall\ z \in \mathbb{C}$. [1]

Let $g(z) = \frac{1}{f(z)}$. Then g is entire and bounded by $\frac{1}{M}$. [1+1]

By Lioiville's Theroem, g is a constant function and hence f is a constant function. [1]

2. (a) Show that the function $f(z) = f(x+iy) = \sqrt{|xy|}$ satisfies Cauchy-Riemann Equations at 0 but it is not differentiable at 0. [5]

Solution. $f(z) = u(x,y) + iv(x,y) = \sqrt{|xy|}$ implies that $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$.

The Cauchy Riemann equations are $u_x = v_y$ and $u_y = -v_x$ at $z_0 = x_0 + iy_0$. Thus,

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = 0. \qquad [1]$$

Similarly, $u_y(0,0) = 0$. Moreover, $v_x(0,0) = 0 = v_y(0,0) = 0$, Hence, the Cauchy-Riemann Equations are satisfied at 0. [1]

Now,

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{(h_1+ih_2)\to 0} \frac{f(h_1+ih_2)}{h_1+ih_2} = \lim_{(h_1+ih_2)\to 0} \frac{\sqrt{h_1h_2}}{h_1+ih_2}. \qquad [1]$$

Approaching 0 through the line $h_2 = m^2h_1$, the above limit becomes $\dfrac{m}{1+im^2}$ [1]

which different for different m. Hence $f(z)$ is not differentiable at 0. [1]

(b) Let $\alpha,\ \beta \in \mathbb{C}$ be such that $|\alpha| < |\beta|$. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (3\alpha^n - 5\beta^n) z^n.$$

[4]

Solution. The radius of convergence R is given by

$$\begin{aligned}
\frac{1}{R} &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} && [1] \\
&= \lim_{n\to\infty} \left| \frac{3\alpha^{n+1} - 5\beta^{n+1}}{3\alpha^n - 5\beta^n} \right| \\
&= \lim_{n\to\infty} \frac{|\beta^{n+1}|}{|\beta^n|} \left| \frac{3(\frac{\alpha}{\beta})^{n+1} - 5}{3(\frac{\alpha}{\beta})^n - 5} \right| && [1] \\
&= |\beta| \quad \left(\because \left|\frac{\alpha}{\beta}\right| < 1 \text{ and } \lim_{n\to\infty} \left(\frac{\alpha}{\beta}\right)^n = 0\right) && [2]
\end{aligned}$$

Hence, the radius of convergence is $\frac{1}{|\beta|}$.

(c) Let P be a polynomial of degree $n \geq 1$ with distinct roots. Let γ be a simple closed curve oriented counter clockwise, which does not pass through any root of P but encloses all roots of P. If P' denotes the derivative of P then find the value of the integral

$$\int_\gamma \frac{P'(z)}{P(z)} dz.$$

[5]

Solution. Let $z_1, z_2, \ldots, z_n$ be zeros of P and $P(z) = \alpha(z - z_1) \cdots (z - z_n)$ where $\alpha \in \mathbb{C}$. [2]

Then $\frac{P'(z)}{P(z)} = \sum_{i=1}^{n} \frac{1}{z - z_i}$. [2]

By Cauchy's integral formula, $\int_\gamma \frac{P'(z)}{P(z)} dz = \sum_{i=1}^{n} \int_\gamma \frac{dz}{z - z_i} = 2\pi i n.$ [1]

(d) Find all possible series expansions of the function $f(z) = \frac{1}{3z - z^2 - 2}$ in those regions which are bounded. [6]

Solution. The function $f(z) = \frac{1}{3z - z^2 - 2} = \frac{1}{z-1} - \frac{1}{z-2}$ [1]
which has two singular points $z = 1$ and $z = 2$, is analytic in the domains

$$|z| < 1, \; 1 < |z| < 2, \; 2 < |z| < \infty$$

in which the domains $D_1 : |z| < 1$ and $D_2 : 1 < |z| < 2$ are bounded. [1]
Since $|z| < 1$ and $|z/2| < 1$ in D_1 we have

$$\begin{aligned}
f(z) &= -\frac{1}{1-z} + \frac{1}{2} \cdot \frac{1}{1 - (z/2)} && [1] \\
&= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \\
&= \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n. && [1]
\end{aligned}$$

In D_2, $1 < |z| < 2$ implies that $|1/z| < 1$ and $|z/2| < 1$. Hence,

$$\begin{aligned} f(z) &= \frac{1}{z} \cdot \frac{1}{1-(1/z)} + \frac{1}{2} \cdot \frac{1}{1-(z/2)} && [1] \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}. && [1] \end{aligned}$$

3. (a) Determine

(i) $\mathcal{L}\left(\frac{1-\cos\omega t}{t}\right)$ mentioning the range for which the Laplace transform exists. [6]

Solution. If f is piecewise continuous on $[0,\infty)$ and of exponential order α, with $F(s) = \mathcal{L}(f(t))$ for real $s > \alpha$, and such that $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists, then [1]

$$\int_s^{\infty} F(\xi)\, d\xi = \mathcal{L}\left(\frac{f(t)}{t}\right) \quad (s > \alpha). \qquad [1]$$

Now,

$$f(t) = 1 - \cos t, \quad t > 0, \text{ is of exponential order } 0.$$

$$F(s) = \mathcal{L}(1-\cos t) = \frac{1}{s} - \frac{s}{s^2+\omega^2}. \qquad [1]$$

Thus, for $s > 0$,

$$\begin{aligned} \mathcal{L}\left(\frac{1-\cos t}{t}\right) &= \lim_{t\to\infty} \int_s^t \left(\frac{1}{\xi} - \frac{\xi}{\xi^2+\omega^2}\right) d\xi && [1] \\ &= \frac{1}{2} \lim_{t\to\infty} \left[\log\left(\frac{\xi^2}{\xi^2+\omega^2}\right)\right]_s^t && [1] \\ &= \frac{1}{2} \log\left(1 + \frac{\omega^2}{s^2}\right). && [1] \end{aligned}$$

(ii) $\mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2-2}\right)$. [4]

Solution. $\frac{e^{-\pi s}}{s^2-2} = e^{-\pi s}\mathcal{L}\left(\frac{1}{\sqrt{2}}\sinh(\sqrt{2}t)\right)$. [2]

Therefore, $\mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2-2}\right) = \frac{1}{\sqrt{2}}\, u_\pi(t)\sinh(\sqrt{2}(t-\pi))$. [2]

(b) Solve the integro-differential equations by the Laplace transform method

$$x'(t) + \int_0^t x(t-\tau)d\tau = \cos t, \qquad x(0) = 0.$$

[15]

Solution. The above integro-differential equation can be rewritten as

$$x'(t) + (x * 1)(t) = \cos t, \qquad x(0) = 0. \qquad [1]$$

As $\cos t$ is continuous and of exponential order and assuming $x(t)$ is continuous and of exponential order, taking Laplace transform both sides we get for $\mathcal{R}e(s) > 0$, [1]

$$\begin{aligned}
\mathcal{L}(x'(t)) + \mathcal{L}((x*1)(t)) &= \frac{s}{s^2+1} && [1] \\
\implies s\mathcal{L}(x(t)) + \mathcal{L}(x(t)) \cdot \mathcal{L}(1) &= \frac{s}{s^2+1} && [2+2] \\
\implies \mathcal{L}(x(t)) = \frac{s^2}{(s^2+1)^2} &= \frac{1}{s^2+1} - \frac{1}{(s^2+1)^2} && [1] \\
\implies x(t) = \sin t &- \mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) && [1]
\end{aligned}$$

Now,

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) &= \mathcal{L}^{-1}\left(\frac{1}{s^2+1} \cdot \frac{1}{s^2+1}\right) && [1] \\
&= \sin t * \sin t && [1] \\
&= \int_0^t \sin\tau \sin(t-\tau) d\tau && [1] \\
&= \frac{1}{2}\int_0^t (-\cos t + \cos\tau) d\tau && [1] \\
&= \frac{1}{2}\left(-t\cos t + \sin t\right). && [1]
\end{aligned}$$

Therefore,

$$x(t) = \frac{1}{2}(\sin t + t\cos t). \qquad [1]$$

4. (a) Determine whether the following statements are true or false and justify your answer.

   i. Let $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \; dx,\ n = 0, 1, \ldots$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \; dx,\ n = 1, 2, \ldots$. Then

   $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).$$

   [3]

   Solution. False. If $f(x)$ is a periodic function with period 2π, piecewise continuous in the interval $[-\pi, \pi]$ and have left-hand and right-hand derivative at each point, then above statement is true [2]
   except at points x_0 where $f(x)$ is discontinuous. The sum of the series at x_0 is $\frac{1}{2}[f(x_0^-) + f(x_0^+)]$. [1]

   ii. The function $f(x) = \cos x$ can be expressed in a Fourier sine series on the interval $-\pi \le x \le \pi$. [2]
   False. As $\cos x$ is an even function and the interval is $-\pi \le x \le \pi$, $b_n = 0$. Hence, the Fourier series of $\cos x$ will be pure cosine series. [2]

(b) Find the Fourier transform of the function

$$f(x) = \begin{cases} -1 & \text{if} \quad -1 < x < 0 \\ 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

[5]

Solution.

$$\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx & [1] \\
&= -\frac{1}{\sqrt{2\pi}} \int_{-1}^{0} e^{-iwx} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{1} e^{-iwx} dx & [1] \\
&= -\frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-iwx}}{-iw}\Big|_{-1}^{0} + \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-iwx}}{-iw}\Big|_{0}^{1} & [1] \\
&= \frac{1}{iw\sqrt{2\pi}} \left(1 - e^{iw} - e^{-iw} + 1\right) & [1] \\
&= \frac{1}{iw\sqrt{2\pi}} (2 - 2\cos w) \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos w}{iw} \right). & [1]
\end{aligned}$$

(c) Find the discrete Fourier transform of $z = (1, i, 2+i, 3)$. [3]

Solution. The discrete Fourier transform of z is given by $\hat{z} = W_4 z$, [1]
where

$$W_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \qquad [1]$$

Thus,

$$\begin{aligned}
\hat{z} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \\ 2+i \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 6+2i \\ 2i \\ 0 \\ -2-4i \end{pmatrix}. & [1]
\end{aligned}$$