Indian Institute of Information Technology Allahabad
Complex Analysis and Integral Transformation (SMAT330)

Back Paper Examination - 2016

Duration: 3 Hour
Full Marks: 75
Instructor: Abdullah Bin Abu Baker

Date: July 19, 2016
Time: 14:00 – 17:00 IST
BTech - III (IT+RGIT)

This question paper contains **two** pages. Attempt **all** the questions. Numbers indicated on the right in [ ] are full marks of that particular question.

Notations:

1. $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $i^2 = -1$.
2. $\mathcal{F}(s) = \mathcal{L}(f(t))$ denotes the Laplace transform of a function f and $\mathcal{L}^{-1}(\mathcal{F}(s)) = f(t)$ denote the inverse Laplace transform.
3. $\mathfrak{F}_{\mathfrak{s}}\{f(x)\}$ and $\mathfrak{F}_{\mathfrak{c}}\{f(x)\}$ denotes the Fourier sine and cosine transforms respectively.

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1. Provide a short proof or answer of the following statements.
   (a) The set $\{z \in \mathbb{C} : \frac{1}{2} < |z| < \frac{7}{3}\}$ is connected. [1]
   (b) Suppose f and g are piecewise continuous functions on $[0, \infty]$, and have exponential order α and β respectively. Then $\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$, where a and b are arbitrary constants. [2]
   (c) Let $f(x)$ be continuous and absolutely integrable on the x-axis, $f'(x)$ piecewise continuous on every finite interval, and $\lim_{x\to\infty} f(x) = 0$. Then
   $$\mathfrak{F}_{\mathfrak{s}}\{f'(x)\} = -w\mathfrak{F}_{\mathfrak{c}}\{f(x)\}.$$
   [3]
   (d) Any bounded function has exponential order. [1]
   (e) Evaluate $\displaystyle\int_{|z-i|=2} \frac{e^z}{z(z+3)}dz$. [4]
   (f) Does there exists a function f such that $\mathcal{L}(f(t)) = \dfrac{s}{\log s}$. [2]
   (g) Lef f be an entire function such that $f(z) > M$, $\forall z \in \mathbb{C}$, for some constant M. Then f is a constant function. [4]

2. (a) Show that the function $f(z) = f(x+iy) = \sqrt{|xy|}$ satisfies Cauchy-Riemann Equations at 0 but it is not differentiable at 0. [5]
   (b) Let $\alpha,\ \beta \in \mathbb{C}$ be such that $|\alpha| < |\beta|$. Find the radius of convergence of the power series
   $$\sum_{n=0}^{\infty}(3\alpha^n - 5\beta^n)z^n.$$
   [4]

(c) Let P be a polynomial of degree $n \geq 1$ with distinct roots. Let γ be a simple closed curve oriented counter clockwise, which does not pass through any root of P but encloses all roots of P. If P' denotes the derivative of P then find the value of the integral

$$\int_\gamma \frac{P'(z)}{P(z)} dz.$$

[5]

(d) Find all possible series expansions of the function $f(z) = \dfrac{1}{3z - z^2 - 2}$ in those regions which are bounded. [6]

3. (a) Determine

(i) $\mathcal{L}\left(\dfrac{1-\cos\omega t}{t}\right)$ mentioning the range for which the Laplace transform exists. [6]

(ii) $\mathcal{L}^{-1}\left(\dfrac{e^{-\pi s}}{s^2-2}\right)$. [4]

(b) Solve the integro-differential equations by the Laplace transform method

$$x'(t) + \int_0^t x(t-\tau)d\tau = \cos t, \qquad x(0) = 0.$$

[15]

4. (a) Determine whether the following statements are true or false and justify your answer.

i. Let $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\ dx,\ n = 0, 1, \ldots$ and $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\ dx,\ n = 1, 2, \ldots$. Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

[3]

ii. The function $f(x) = \cos x$ can be expressed in a Fourier sine series on the interval $-\pi \leq x \leq \pi$. [2]

(b) Find the Fourier transform of the function

$$f(x) = \begin{cases} -1 & \text{if } \ -1 < x < 0 \\ 1 & \text{if } \quad 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

[5]

(c) Find the discrete Fourier transform of $z = (1, i, 2+i, 3)$. [3]