A study of some classes of operators on Banach spaces

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Abstract

In this thesis, we propose to study two classes of operators on Banach spaces. One is the class of *'local isometries'*, and the second is the class of *'projections'* which are related to isometries.

In the first part of the thesis, we characterize local isometries on strongly separating subspaces of $C_0(X)$, X being a locally compact Hausdorff space, and weakly normal closed subalgebras of $C_u(K_E)$, the Banach algebra of all real or complex-valued uniformly continuous bounded functions defined on K_E endowed with the supremum norm, where K_E is a closed subset of the Banach space E. We prove that under some conditions on the subspaces and subalgebras, any local isometry is a global isometry, that is, a surjective linear isometry. We prove similar results for local isometries on $C^2[0,1]$, the Banach space of all (real or complex valued) functions that have continuous derivatives f', f'' on the closed unit interval $[0,1]$, equipped with norm $\|f\| = |f(0)| + |f'(0)| + \|f''\|_\infty$, where $\|\cdot\|_\infty$ is the usual supremum norm. We further find out the structure of isometries of finite order on $C^2[0,1]$, and extend the above results for the class of local isometries of finite order as well.

In the second part of the thesis, we study norm-one projections on $C^2[0,1]$. We characterize projections on $C^2[0,1]$ that can be written as convex combination of two surjective linear isometries. We also determine the structure of Hermitian projections and generalized bi-circular projections on $C^2[0,1]$. At the end, we discuss the relationship of these two types of projections (Hermitian and generalized bi-circular projections) with the convex combination of two isometries.

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January, 2022 **Rahul Maurya**

Dedicated

to

My Family

CHAPTER 1

Introduction

In this chapter, we give a brief history and motivation of the problems which we attempt to study in this thesis. We also introduce few notations and recall some definitions and results which will be used later. At the end, we provide a chapter-wise summary of the main results.

1.1 History

The study of transformations preserving lengths and angles has been a central theme in mathematics for a long time. A transformation which preserves the distance between every two elements of the space (metric space, normed linear space) is called an isometry. Some examples of isometries on Euclidean spaces are translations, rotations, and reflections; another example is the Fourier transform on $L^2(\mathbb{R})$. Many properties of an isometric transformation, like injectivity and continuity etc., can be easily obtained from the distance preserving criterion. Moreover, if a transformation on a normed linear space is linear, then being an isometry is equivalent to being norm preserving.

The study of isometries on Banach spaces plays an important role to understand its structure and geometry. Stefan Banach was the first to raise the question concerning the structure of a surjective linear isometry on a specific Banach space. In his treatise [11] he described the form of isometries on $C(X)$, where X is a compact metric space, and $L^p[0,1]$, $(1 \leq p < \infty,\ p \neq 2)$. At the same time, researchers began investigating the isometries of other Banach spaces as well. One of the most classical results in this area is the Banach-Stone theorem describing surjective linear isometries on $C_0(X)$, X being a locally compact Hausdorff space. This classical theorem has been generalized by many authors in several directions, for example, by considering into linear isometries, replacing $C_0(X)$ by its subspaces and subalgebras, or by looking into Banach spaces of vector-valued functions. A comprehensive account of this theory can be found in the monographs by Behrends [12], and by Fleming and Jamison [27, 28].

Another class of transformations which is crucial in understanding the structure of a Banach space is the class of projections. Simple examples of projections on Euclidean spaces are idempotent matrices. A standard result of linear algebra says that every diagonalizable matrix can be decomposed as a linear sum of idempotent matrices. The spectral theory of operators demonstrates that projections appear as basic building blocks of more complicated operators. Attempt to describe projections with desired properties (for example, norm-one projections) has received lot of attention in past as well as in recent times.

One of the problems posed by Banach [11] about projections is whether every Banach space E admits a non-trivial projection? A non-trivial projection here means a projection P on E such that $\dim P(E) = \infty$ and $\dim E/P(E) = \infty$. This question was answered negatively by Gowers and Maurey in 1993 [31]. We note that any Banach space can be equivalently renormed so that the given projection has norm one. Intensive studies were carried out by researchers in the last century to characterize norm-one projections on classical Banach spaces. The survey article by Randrianantoanina [50] is an excellent reference for numerous results on norm-one projections.

1.2 Motivation

Let E be a Banach space. We respectively denote by $B(E)$ and $\mathcal{G}(E)$, the Banach space of all bounded linear operators, and the set of all surjective linear isometries, on E. Let $T \in B(E)$ such that for every $x \in E$, Tx coincides with the action of a surjective linear isometry on x, that is, there exists a $T_x \in \mathcal{G}(E)$ (depending on x, that is why the subscript x and this isometry may vary from point to point) such that $T(x) = T_x(x)$. One may ask under what conditions $T \in \mathcal{G}(E)$. Such a T is called a local surjective isometry and we say that T interpolates $\mathcal{G}(E)$. We observe here that any local surjective isometry is in fact an isometry. Indeed, $||T(x)|| = ||T_x(x)|| = ||x||$. So, the problem reduces to see whether any local surjective isometry is surjective?

Since any injective linear map on a finite dimensional vector space is surjective, the problem stated in the previous paragraph has a positive answer in the case of finite dimensional Banach spaces. Now, let E be an infinite dimensional Hilbert space, and T be any into isometry on E. Let $x, y \in E$ such that $Tx = y$. As $||x|| = ||y||$, there exists an operator $S \in \mathcal{G}(E)$ such that $S(x) = y$. Therefore, T is a local surjective isometry on E which is not surjective. Thus, the problem has a negative answer if the space under consideration is an infinite dimensional Hilbert space.

It is, therefore, a natural question to ask what happens in other infinite dimensional Banach spaces. This is a very basic problem in the sense that we want to get a global conclusion from a local hypothesis.

Besides the isometry group, the above problem can be studied for other important classes of transformations on operator algebras like automorphism group and derivations. The problem then would be to see whether any local automorphism (derivation) is an automorphism (a derivation). Investigations of this kind were initiated by Kadison, Larson and Sourour [38, 41, 42]. The book by Molnár [47] is a pertinent reference for the study of local maps on operator algebras and function algebras.

We consider the following definition.

Definition 1.2.1. *Let $\mathcal{S} \subset B(E)$. We define the algebraic closure of $\mathcal{S}$ as*

$$\overline{\mathcal{S}}^a = \{T \in B(E) : Tx \in \mathcal{S}x, \ \forall \ x \in E\},$$

where $\mathcal{S}x = \{Sx : S \in \mathcal{S}\}$.

Clearly, $\mathcal{S} \subseteq \overline{\mathcal{S}}^a$. The subset $\mathcal{S}$ is called algebraically reflexive if $\mathcal{S} = \overline{\mathcal{S}}^a$.

Elements of the algebraic closure of $\mathcal{S}$ are called local maps. It is clear that $\mathcal{S}$ is algebraically reflexive if for every map T that belongs locally to $\mathcal{S}$, we necessarily have $T \in \mathcal{S}$.

If $\mathcal{S} = \mathcal{G}(E)$, then $\mathcal{G}(E)$ is called algebraically reflexive if $\mathcal{G}(E) = \overline{\mathcal{G}(E)}^a$, that is, every local surjective isometry is surjective. We recall that a local isometry is an isometry. From the preceding discussion, we see that, if E is finite dimensional, then $\mathcal{G}(E)$ is algebraically reflexive, and if E is an infinite dimensional Hilbert space, then $\mathcal{G}(E)$ fails to be algebraically reflexive.

A natural setting for studying the algebraic reflexivity of the isometry group of a Banach space is where a complete description is available. This is the case in most of the well known Banach spaces. In the last decades, a lot of work has been done in this direction, see for instance [22, 37, 44, 45, 49] and [51]. In [44], Molnár and Zalar proved that $\mathcal{G}(c_0)$ and $\mathcal{G}(\ell_p)$ $(1 \leq p \leq \infty, \ p \neq 2)$ are algebraically reflexive. In the same paper it was also proved that, for a first countable compact Hausdorff space X, $\mathcal{G}(C(X))$ is algebraically reflexive. Similar results were extended in [23] to other classical Banach spaces, such as spaces of measurable functions, Hardy spaces, Banach algebras of holomorphic functions.

A surjective linear isometry $T \in B(E)$ is called an isometry of finite order if there exists $n \in \mathbb{N}$ such that $T^n = I$, where I denotes the identity operator on E. For $n \in \mathbb{N}$, let $\mathcal{G}^n(E) = \{T \in \mathcal{G}(E) : T^n = I\}$. An operator $T \in \mathcal{G}^n(E)$ is called an isometry of order n. Dutta and Rao [25] proved that for a compact Hausdorff space Ω, if $\mathcal{G}(C(X))$ is algebraically reflexive, then $\mathcal{G}^2(C(X))$ is also algebraically reflexive. This result was generalized in [5] to isometries of order n on $C_0(X, E)$, where X is a first countable locally compact Hausdorff space, and E is a Banach space having the strong Banach-Stone property.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be nonzero complex numbers, and $T_1, T_2, \ldots, T_n$ be operators in $B(E)$. It is a natural question to ask when $P = \alpha_1 T_1 + \alpha_2 T_2 + \cdots + \alpha_n T_n$ is a projection? This question has been partially answered by several authors recently. In [2], the authors proved that if T is an operator of order n, i.e., $T^n = I$, then $P = \sum_{i=0}^{n-1} \alpha_i T^i$ is a projection if and only if $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ is the inverse discrete Fourier transform of δ_S, for some $S \subseteq \{0, 1, \ldots, n-1\}$, where δ_S is the vector with components given by $\delta_S(i) = 1$ for $i \in S$ and $\delta_S(i) = 0$ otherwise. Botelho [16] studied the properties of operators that are in the convex hull of a finite set of surjective isometries on the space $C(X)$, where X is a compact connected Hausdorff topological space. The same problem has been studied for other Banach spaces, like minimal norm ideals, spaces of analytic functions, Hardy spaces, absolutely continuous function spaces, etc., see [1, 20, 21, 32, 34] and [40].

Definition 1.2.2. *A projection P on a Banach space E is said to be a generalized bi-circular projection if there exists an $\alpha \in \mathbb{T} \setminus \{1\}$ such that $P + \alpha(I - P)$ is an isometry on E. Here, $\mathbb{T}$ denotes the unit circle in the complex plane.*

The notion of generalized bi-circular projections was introduced in 2007 by Fošner, Ilišević, and Li [30]. In [30], the authors characterized generalized bi-circular projections on finite dimensional Banach spaces with respect to various G-invariant norms. After that, this class of projections were extensively studied by several authors for many Banach spaces, see for example, [18, 33, 43], and the references therein.

It was shown in [43] that any generalized bi-circular projection is bicontractive, i.e., a projection P such that $||P|| = ||I - P|| = 1$. Moreover, any projection on a Hilbert space is a generalized bi-circular projections if and only if it is an orthogonal projection (see [19, Proposition 3.1]).

1.3 Research objectives

In this thesis, we propose to study two classes of operators on Banach spaces. One is the class of *'local isometries'*, and the second is the class of *'projections'* which are related to isometries.

For a Banach space E and a closed subset K_E of E, we denote by $C_u(K_E)$ the Banach algebra of all real or complex-valued, uniformly continuous bounded functions defined on K_E endowed with the supremum norm.

Definition 1.3.1. *Let A be a subspace of $C_0(X)$. We say that A is strongly separating if given any pair of distinct points x_1, x_2 of X, there exists $f \in A$ such that $|f(x_1)| \neq |f(x_2)|$.*

Definition 1.3.2. *A closed subalgebra $A_u(K_E)$ of $C_u(K_E)$ is said to be weakly normal if given any subsets A and B of K_E with a positive distance $d(A, B) = \inf\{||a - b|| : a \in A, b \in B\}$, there is an $f \in A_u(K_E)$ such that $|f(x)| \geq 1$ for every $x \in A$, and $|f(y)| \leq \frac{1}{2}$ for every $y \in B$.*

For a weakly normal closed subalgebra $A_u(K_E)$ of $C_u(K_E)$, we denote by $A_u^0(K_E)$ the subalgebra of $A_u(K_E)$ whose elements vanishes at $0 \in K_E$, that is,

$$A_u^0(K_E) = \{f \in A_u(K_E) : f(0) = 0\}.$$

In the first part of the thesis, we study the problem of algebraically reflexivity of the following sets:

1. The set of all surjective linear isometries between strongly separating subspaces of $C_0(X)$.
2. The set of all surjective linear isometries between weakly normal closed subalgebras of $C_u(K_E)$.
3. The set of all surjective linear isometries between subalgebras of $A_u(K_E)$ whose elements vanish at 0.
4. The group of all surjective linear isometries on the space of 2-times continuously differentiable functions.
5. The set of all surjective linear isometries of finite order on the space of 2-times continuously differentiable functions.

The remaining part of the thesis is focussed in studying projections on the space of 2-times continuously differentiable functions.

Definition 1.3.3. *An operator $T \in B(X)$ is said to be Hermitian if $e^{i\theta T}$ is an isometry for every $\theta \in \mathbb{R}$.*

Hermitian operators on various complex Banach spaces were investigated by many authors, see for example [13, 14, 15, 17] and [26].

On the space of 2-times continuously differentiable functions, we attempt to address the following problems:

1. Characterize projections that can be written as convex combination of two surjective linear isometries.

2. Find out the structure of Hermitian and generalized bi-circular projections.

3. Study the relationship of Hermitian and generalized bi-circular projections with the convex combination of two isometries.

1.4 Preliminaries and basic results

In this section, we introduce some notations and recall some definitions and results that will be used throughout this thesis.

We shall assume E and F to be Banach spaces. We respectively denote by $B(E, F)$ and $\mathcal{G}(E, F)$, the Banach space of all bounded linear operators, and the set of all surjective linear isometries, from E to F. If $E = F$, then $B(E, E)$ is denoted by $B(E)$, and $\mathcal{G}(E, E)$ by $\mathcal{G}(E)$.

Let $\mathbb{K}$ denotes the field of real or complex numbers. We denote by $C_0(X)$, the space of all $\mathbb{K}$-valued continuous functions on a locally compact Hausdorff space X vanishing at infinity. We recall that a continuous function $f : X \to \mathbb{K}$ is said to vanish at infinity if for all $\varepsilon > 0$, the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact.

Definition 1.4.1. *Let A be a subspace of $C_0(X)$. A subset U of X is said to be a boundary for A if each function in A attains its maximum on U. The Shilov boundary of A, denoted ∂A, is the unique minimal closed boundary for A.*

The structure of into and onto linear isometries of a strongly separating subspace of $C_0(X)$ into $C_0(Y)$ are given in the next two theorems.

Theorem 1.4.2. *[9, Theorem 3.1] Let T be a linear isometry of a strongly separating linear subspace A of $C_0(X)$ into $C_0(Y)$. Then there are a subset Y_0 of Y, which is a boundary for $T(A)$, a continuous map h from Y_0 onto $\sigma_0 A$ and a continuous map $a : Y_0 \to \mathbb{K}$, such that $|a(y)| = 1$ for all $y \in Y_0$, and*

$$Tf(y) = a(y)f(h(y)) \text{ for all } y \in Y_0 \text{ and all } f \in A.$$

Furthermore, if $\sigma_0 A$ is compact, then Y_0 is closed.

Theorem 1.4.3. *[9, Theorem 4.1] Let T be a linear isometry of a strongly separating linear subspace A of $C_0(X)$ onto such a subspace B of $C_0(Y)$. Then there exist a homeomorphism h of $\sigma_0 B$ onto $\sigma_0 A$ and a continuous map $a : \sigma_0 B \to \mathbb{K}$, such that $|a(y)| = 1$ for all $y \in \sigma_0 B$, and*

$$Tf(y) = a(y)f(h(y)) \text{ for all } y \in \sigma_0 B \text{ and all } f \in A.$$

The next two theorems characterize surjective linear isometries on the subalgebras $A_u(K_E)$ and $A_u^0(K_E)$ of $C_u(K_E)$.

Theorem 1.4.4. *[10, Theorem 4.3] Let X and Y be Banach spaces and let $T : A_u(K_X) \to A_u(K_Y)$ be a linear surjective isometry. Then there exist a uniform homeomorphism h of K_Y onto K_X, and a function $a \in C_u(K_Y)$, such that $|a(y)| = 1$ for all $y \in K_Y$, and $Tf(y) = a(y)f(h(y))$ for all $y \in K_Y$ and for all $f \in A_u(K_X)$.*

Theorem 1.4.5. *[10, Theorem 4.8] Let X and Y be Banach spaces and let $T : A_u^0(K_X) \to A_u^0(K_Y)$ be a linear surjective isometry. Then there exists a uniform homeomorphism h*

of K_Y onto K_X with $h(0) = 0$. Furthermore, there is a function $a \in C(K_Y \setminus \{0\})$, with $|a(y)| = 1$ for all $y \in K_Y \setminus \{0\}$, such that, for all $f \in A_u^0(K_X)$

$$Tf(y) = \begin{cases} a(y)f(h(y)), & y \in K_Y \setminus \{0\}, \\ 0, & y = 0. \end{cases}$$

For each nonnegative integer r, let $C^r[0,1]$ be the space of all (real or complex valued) functions that have continuous derivatives $f^{(1)}, f^{(2)}, \dots, f^{(r)}$ of order upto r on the closed unit interval $[0,1]$. Let

$$||f|| = \sum_{i=0}^{r-1} |f^{(i)}(0)| + ||f^{(r)}||_\infty,$$

where $||\cdot||_\infty$ denotes the usual supremum norm. The space $C^r[0,1]$ is a Banach space with the norm $||\cdot||$. For $r = 0$, $C^r[0,1]$ is simply denoted by $C[0,1]$, the space of all (real or complex valued) continuous functions on $[0,1]$ with the supremum norm. Moreover, we shall denote the first and second derivatives of f by f' and f'', respectively.

The structure of surjective linear isometries on $C^r[0,1]$ is given in the next theorem. Define a map $\zeta : C[0,1] \to C^1[0,1]$ by $(\zeta f)(x) = \int_0^x f(s)ds$. It is clear by the fundamental theorem of calculus that $(\zeta f)'(x) = f(x)$ for every $x \in [0,1]$.

Theorem 1.4.6. *[39, Theorem 1.1] Let T be linear operator from $(C^r[0,1], \|\cdot\|)$ onto itself. Then T is an isometry if and only if there exist a homeomorphism ϕ of $[0,1]$ onto itself, a unimodular continuous function ω on $[0,1]$, a permutation $\{\tau(0), \tau(1), \dots, \tau(r-1)\}$ of $\{0, 1, \dots, r-1\}$ and unimodular constants $\lambda_0, \lambda_1, \dots, \lambda_{r-1}$ such that*

$$Tf(x) = \sum_{i=0}^{r-1} \frac{\lambda_i f^{(\tau(i))}(0)}{i!} x^i + (\zeta^r(\omega(f^{(r)} \circ \phi)))(x).$$

The next theorem is simply a statement of the above theorem for the space $C^2[0,1]$. Let $\mathbb{T}$ denotes the unit circle in the complex plane.

Theorem 1.4.7. *A surjective linear operator $T : C^2[0,1] \to C^2[0,1]$ is an isometry if and only if there exist a homeomorphism $\phi : [0,1] \to [0,1]$, a continuous function $\omega : [0,1] \to \mathbb{T}$ and constants $\lambda, \mu \in \mathbb{T}$ such that either*

$$Tf(x) = \lambda f(0) + \mu f'(0)x + (\zeta^2(\omega(f'' \circ \phi)))(x), \tag{$\star$}$$

or

$$Tf(x) = \lambda f'(0) + \mu f(0)x + (\zeta^2(\omega(f'' \circ \phi)))(x). \tag{$\star\star$}$$

Remark 1.4.8. *An isometry of the form* ($\star$) *will be referred as an isometry of the first type. Similarly, an isometry of the form* ($\star\star$) *will be referred as an isometry of the second type.*

Definition 1.4.9. 1. *A projection P on a Banach space E is said to be a generalized bi-circular projection, (GBP, for short), if there exists an $\alpha \in \mathbb{T} \setminus \{1\}$ such that $P + \alpha(I - P)$ is an isometry on E.*

2. *A projection P on a Banach space E is said to be a bi-circular projection if $P + \alpha(I - P)$ is an isometry on E for all $\alpha \in \mathbb{T}$.*

Remark 1.4.10. *If P is a GBP on a Banach space E, then there exist $\alpha \in \mathbb{T} \setminus \{1\}$ and $T \in \mathcal{G}(E)$ such that $P + \alpha(I - P) = T$. The isometry T will be referred as the isometry associated with P.*

1.5 Outline of the thesis

We now give a chapter-wise summary and statement of the main results proved in this thesis.

Chapter 2

In this chapter, we investigate the algebraic reflexivity of the set all surjective linear isometries between certain subspaces of $C_0(X)$, and between various subalgebras of $C_u(K_E)$.

We prove the following results.

Theorem 1.5.1. *Let X and Y be locally compact Hausdorff spaces, and let A and B be strongly separating linear subspaces of $C_0(X)$ and $C_0(Y)$ respectively. If there exists a nonnegative real-valued injective function $g \in A$ and $\sigma_0 A$ is compact, then $\mathcal{G}(A, B)$ is algebraically reflexive.*

Proposition 1.5.2. *Let E and F be Banach spaces. If there exists an injective map $g \in A_u(K_E)$ such that $g(x) \geq 1$ for all $x \in K_E$, then $\mathcal{G}(A_u(K_E), A_u(K_F))$ is algebraically reflexive.*

Proposition 1.5.3. *Let E and F be Banach spaces. If there exists a nonnegative real-valued injective function $g \in A_u^0(K_E)$, then $\mathcal{G}(A_u^0(K_E), A_u^0(K_F))$ is algebraically reflexive.*

Chapter 3

In this chapter, we establish the algebraic reflexivity of the group of isometries on $C^2[0,1]$. We also find out the structure of isometries of finite order on $C^2[0,1]$, and study the algebraic reflexivity of the collection of all such isometries. The structure of surjective linear isometries on $C^2[0,1]$ is given in theorem 1.4.7.

The main result of this chapter is the following.

Theorem 1.5.4. *$\mathcal{G}(C^2[0,1])$ is algebraically reflexive.*

We also prove that

Proposition 1.5.5. *If n is odd, then $\mathcal{G}^n(C^2[0,1])$ is algebraically reflexive.*

Chapter 4

In chapter 4, we describe projections in the convex hull of two isometries in $C^2[0,1]$. To investigate when a convex combination of two isometries on a Banach space is a projection, we will see that it is enough to consider the average of these isometries.

Let $T_1, T_2 \in \mathcal{G}(C^2[0,1])$. Let us denote the scalars λ, μ and the maps ϕ, ω associated with T_1 and T_2 in Theorem 1.4.7 by λ_1, μ_1, ϕ_1, ω_1 and λ_2, μ_2, ϕ_2, ω_2 respectively. We prove the following result.

Theorem 1.5.6. *Let T_1 and T_2 be surjective linear isometries on $C^2[0,1]$. Then $P = \frac{1}{2}(T_1 + T_2)$ is a projection if and only if the following assertions holds:*

1. *The scalars λ_i, μ_i, $i = 1, 2$, satisfy one of the following conditions*

a) $\lambda_1 = -\lambda_2$ *or* $\lambda_1 = \lambda_2 = 1$*;* $\mu_1 = -\mu_2$ *or* $\mu_1 = \mu_2 = 1$.

b) $\lambda_1 = \mu_1 = 1$ *and* $\lambda_2\mu_2 = 1$*;*

2. *Every* $x \in [0, 1]$ *satisfies one of the following statements*

   a) $x = \phi_1(x) = \phi_2(x)$ *and* $\omega_1(x) = \omega_2(x) = 1$.

   b) $\phi_1(x) = \phi_2(x)$ *and* $\omega_1(x) = -\omega_2(x)$.

   c) For $i, j = 1, 2$, $i \neq j$, $x = \phi_i(x) \neq \phi_j(x)$, $\phi_j^2(x) = x$, $\phi_i \circ \phi_j(x) = \phi_j(x)$, $\omega_i(x) = \omega_i(\phi_j(x)) = 1$ *and* $\omega_j(x)\omega_j(\phi_j(x)) = 1$.

We also characterize generalized bi-circular projections and Hermitian projections in the following two results.

Theorem 1.5.7. *An operator* P *on* $C^2[0, 1]$ *is a generalized bi-circular projection if and only if one of the following assertions holds:*

1. *If the isometry* T *associated with* P *is of the first type, then either*

   a) $P = \frac{I+T}{2}$, $\lambda = \pm 1$, $\mu = \pm 1$, *and for every* $x \in [0, 1]$, $\omega(x)\omega(\phi(x)) = 1$, $\phi^2(x) = x$, *or*

   b) P *is Hermitian. In this case,* $\lambda = \alpha$ *or* 1, $\mu = \alpha$ *or* 1.

2. *If the isometry* T *associated with* P *is of the second type, then* $P = \frac{I+T}{2}$, $\lambda\mu = 1$, *and for every* $x \in [0, 1]$, $\omega(x)\omega(\phi(x)) = 1$, $\phi^2(x) = x$.

Corollary 1.5.8. *A projection* P *on* $C^2[0, 1]$ *is Hermitian if and only if there exist real constants* a, b *and* c *such that* $Pf(x) = af(x) + bf(0) + cf'(0)x$ *for all* $f \in C^2[0, 1]$ *and* $x \in [0, 1]$, *where* $(a, b, c) \in \{(1, -1, -1), (1, -1, 0), (1, 0, -1), (1, 0, 0), (0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1)\}$.

In the end of this chapter, we prove that

Theorem 1.5.9. *The average of two surjective linear isometries* T_1 *and* T_2 *on* $C^2[0, 1]$ *is a projection* P *if and only if either* P *is Hermitian or* $P = \frac{I+T}{2}$, *for some isometric reflection* T.

Chapter 5

Finally, chapter 5 discusses the conclusion of the work done, and proposes some research problems for future work.

CHAPTER 2

Local isometries on subspaces and subalgebras of function spaces

In this chapter, we study the structure of local isometries on subspaces of $C_0(X)$, and various subalgebras of $C_u(K_E)$. Our results reply on the assumption that these subspaces and subalgebras support an injective function.

The results of this chapter are from [8].

2.1 Local isometries on strongly separating subspaces of $C_0(X)$

In this section, we prove that any local isometry on strongly separating subspaces of $C_0(X)$ is a surjective isometry. In other words, we prove that the set of all surjective linear isometries on strongly separating subspaces of $C_0(X)$ is algebraically reflexive.

Remark 2.1.1. *Let us define the sets*

$\sigma A = \{x_0 \in X :$ *for each neighbourhood* U *of* x_0, $\exists\, f \in A$ *such that* $|f(x)| \leq \|f\|, \forall\, x \in X - U\}$*, and* $\sigma_0 A = \sigma A \cap \{x \in X : \exists\, f \in A$ *with* $f(x) \neq 0\}$*. It is known that if* A *is a* *subspace of* $C_0(X)$*, then* $\partial A = \sigma A$ *[9, Lemma 2.1].*

Our main result is the following.

Theorem 2.1.2. *Let* X *and* Y *be locally compact Hausdorff spaces, and let* A *and* B *be strongly separating linear subspaces of* $C_0(X)$ *and* $C_0(Y)$ *respectively. If there exists a nonnegative real-valued injective function* $g \in A$ *and* $\sigma_0 A$ *is compact, then* $\mathcal{G}(A, B)$ *is algebraically reflexive.*

Proof. Let $T \in \overline{\mathcal{G}(A, B)}^a$. Since T is an into isometry, Theorem 1.4.2 implies that there exist a subset Y_0 of Y, a continuous onto map $h : Y_0 \to \sigma_0 A$ and a continuous map $\tau : Y_0 \to \mathbb{K}$, such that $|\tau(y)| = 1 \ \forall \ y \in Y_0$, and

$$Tf(y) = \tau(y) f(h(y)), \ \forall \ y \in Y_0 \text{ and } f \in A. \tag{2.1.1}$$

To prove the surjectivity of T we will show that h is a homeomorphism and $Y_0 = \sigma_0 B$.

First we show that h is injective. For the map g given in the hypothesis, there exists $T_g \in \mathcal{G}(A, B)$ such that $Tg = T_g g$. Theorem 1.4.3 implies the existence of a homeomorphism $h_g : \sigma_0 B \to \sigma_0 A$ and a continuous map $\tau_g : \sigma_0 B \to \mathbb{K}$, such that $|\tau_g(y)| = 1 \ \forall \ y \in \sigma_0 B$, and

$$Tg(y) = \tau_g(y) g(h_g(y)), \ \forall \ y \in \sigma_0 B. \tag{2.1.2}$$

From the proof of Theorem 1.4.3 we know that $Y_0 \subseteq \sigma_0 B$. Now, Equations (2.1.1) and (2.1.2) imply that $g(h(y)) = g(h_g(y))$, $\forall \ y \in Y_0$. Thus, $h = h_g$ on Y_0 and hence h is injective. Using [48, Theorem 26.6] we conclude that h is a homeomorphism.

It remains to prove that $\sigma_0 B \subseteq Y_0$. Indeed, for $y \in \sigma_0 B$, we have $h_g(y) \in \sigma_0 A$. As h is onto, there exists $y_0 \in Y_0$ such that $h(y_0) = h_g(y)$. But $h = h_g$ on Y_0, therefore, $y = y_0$. This completes the proof. $\square$

2.2 Local isometries on various subalgebras of $C_u(K_E)$

In this section, we prove the algebraic reflexivity of the set of all surjective linear isometries on weakly normal subalgebras of $C_u(K_E)$ and on the subalgebra $A_u^0(K_E)$.

The following remark is crucial in our proofs.

Remark 2.2.1. *1. The closed subalgebras $A_u(K_E)$ and $A_u^0(K_E)$ can be identified respectively with closed subalgebras $A(E)$ and $A_0(E)$ of $C(\gamma E)$, where γE is a compactification of K_E defined as the quotient space $\gamma E := \beta K_E/\mathcal{R}$. Here, βK_E is the Stone-Cech compactification of K_E, and $\mathcal{R}$ is the equivalence relation defined as $x_1 \mathcal{R} x_2$ if $f(x_1) = f(x_2)$ for every $f \in A_u(K_E)$. It is known that $A(E)$ and $A_0(E)$ separate strongly the points of γE. Moreover, $K_E \subseteq \partial A(E)$ and $\partial A(E) = \gamma E$. Furthermore, $K_E \setminus \{0\} \subseteq \partial A_0(E)$ and $\partial A_0(E) \setminus \{0\} = \gamma E \setminus \{0\}$. More details can be found in [10].*

2. We also note that $A(E)$ is a uniform algebra, that is, a closed separating subalgebra of $C(\gamma E)$ which contains the constants. This means that $A(E)$ is nowhere vanishing, that is, for every $\xi \in \gamma E$, there exists $f \in A(E)$ such that $f(\xi) \neq 0$. It follows from Remark 2.1.1 that

$$\begin{aligned}
\sigma_0 A(E) &= \sigma A(E) \cap \{\xi \in \gamma E : \exists\ f \in A(E) \text{ with } f(\xi) \neq 0\} \\
&= \partial A(E) \cap \gamma(E) \text{ (since } A(E) \text{ is nowhere vanishing)} \\
&= \partial A(E) \\
&= \gamma E.
\end{aligned}$$

3. Further, since $A_0(E)$ strongly separates points of γE, for $\xi \in \gamma E$ such that $\xi \neq 0$, there exists an $f \in A_0(E)$ such that $|f(\xi)| \neq |f(0)|$. Since $f(0) = 0$, we have $|f(\xi)| \neq 0$, and hence, $f(\xi) \neq 0$. Therefore, $\{\xi \in \gamma E : \exists\ f \in A_0(E) \text{ with } f(\xi) \neq 0\} = \gamma E \setminus \{0\}$. This implies that $\sigma_0 A_0(E) = \gamma E \setminus \{0\}$.

Proposition 2.2.2. *Let E and F be Banach spaces. If there exists an injective map $g \in A_u(K_E)$ such that $g(x) \geq 1$ for all $x \in K_E$, then $\mathcal{G}(A_u(K_E), A_u(K_F))$ is algebraically reflexive.*

Proof. Let $T \in \overline{\mathcal{G}(A_u(K_E), A_u(K_F))}^a$. Using Remark 2.2.1 and Theorem 2.1.2, we conclude that T is a surjective linear isometry between closed subalgebras $A(E)$ and $A(F)$ of $C(\gamma E)$ and $C(\gamma F)$ respectively. Theorem 1.4.3 implies the existence of a homeomorphism h :

$\gamma F \to \gamma E$ and a continuous map $\tau : \gamma F \to \mathbb{K}$, such that $|\tau(y)| = 1$ for all $y \in \gamma F$, and

$$Tf(y) = \tau(y)f(h(y)), \ \forall \ y \in \gamma F \text{ and } f \in A(E). \tag{2.2.1}$$

In order to prove that $T : A_u(K_E) \to A_u(K_F)$ is a surjective linear isometry we need to show that $h : K_F \to K_E$ is a uniform homeomorphism and $\mu = \tau|_{K_F} \in C_u(K_F)$.

For the first part, considering the map g given in the hypothesis, there exits $T_g \in \mathcal{G}(A_u(K_E), A_u(K_F))$ such that $Tg = T_g g$. Now, Theorem 1.4.4 implies the existence of a uniform homeomorphism $h_g : K_Y \to K_X$ and a function $\tau_g \in C_u(K_F)$, such that $|\tau_g(y)| = 1$ for all $y \in K_F$, and

$$Tg(y) = \tau_g(y)g(h_g(y)), \ \forall \ y \in K_F. \tag{2.2.2}$$

Comparing Equations (2.2.1) and (2.2.2) and using the injectivity of g we conclude that $h = h_g$ on K_F. Thus, h is a uniform homeomorphism.

To prove the second part, suppose on the contrary that μ is not uniformly continuous on K_F. Then there exist $\varepsilon > 0$ and two sequences (x_n) and (y_n) in K_F such that $\lim_{n\to\infty} ||x_n - y_n|| = 0$ and $|\mu(x_n) - \mu(y_n)| \geq \varepsilon$ for every $n \in \mathbb{N}$.

As Tg is uniformly continuous, $\lim_{n\to\infty}(Tg(x_n) - Tg(y_n)) = 0$ or

$$\lim_{n\to\infty}(\mu(x_n)g(h(x_n)) - \mu(y_n)g(h(y_n))) = 0. \tag{2.2.3}$$

Similarly for the map g^2 we will have

$$\lim_{n\to\infty}(\mu(x_n)g^2(h(x_n)) - \mu(y_n)g^2(h(y_n))) = 0. \tag{2.2.4}$$

Multiplying Equation (2.2.3) by $g(h(x_n))$ we get

$$\lim_{n\to\infty}(\mu(x_n)g^2(h(x_n)) - \mu(y_n)g(h(x_n))g(h(y_n))) = 0. \tag{2.2.5}$$

Subtracting Equations (2.2.4) and (2.2.5) we will get

$$\lim_{n\to\infty}(\mu(y_n)g^2(h(y_n)) - \mu(y_n)g(h(x_n))g(h(y_n))) = 0.$$

This implies that

$$\lim_{n\to\infty}(g(h(x_n)) - g(h(y_n))) = 0. \tag{2.2.6}$$

Lastly multiplying Equation (2.2.6) by $\mu(x_n)$ and subtracting Equation (2.2.3) we get

$$\lim_{n\to\infty} g(h(y_n))(\mu(x_n) - \mu(y_n)) = 0.$$

This is a contradiction. Hence, μ is uniformly continuous on K_F and the proof is complete. □

Proposition 2.2.3. *Let E and F be Banach spaces. If there exists a nonnegative real-valued injective function $g \in A_u^0(K_E)$, then $\mathcal{G}(A_u^0(K_E), A_u^0(K_F))$ is algebraically reflexive.*

Proof. Let $T \in \overline{\mathcal{G}(A_u^0(K_E), A_u^0(K_F))}^a$. Using the same arguments of Proposition 2.2.2, there exist a homeomorphism $h : \gamma F\setminus\{0\} \to \gamma E\setminus\{0\}$ and a continuous map $\tau : \gamma F\setminus\{0\} \to \mathbb{K}$, such that $|\tau(y)| = 1$ for all $y \in \gamma F \setminus \{0\}$, and

$$Tf(y) = \tau(y)f(h(y)),\ \forall\ y \in \gamma F \setminus \{0\} \text{ and } f \in A_0(E). \tag{2.2.7}$$

Since $\tau|_{K_F\setminus\{0\}} \in C(K_F \setminus \{0\})$, in order to prove that $T : A_u^0(K_E) \to A_u^0(K_F)$ is a surjective linear isometry we just need to show that $h : K_F \to K_E$ is a uniform homeomorphism with $h(0) = 0$.

For the map g in the hypothesis, there exists $T_g \in \mathcal{G}(A_u^0(K_E), A_u^0(K_F))$ such that $Tg = T_g g$. Theorem 1.4.5 implies the existence of a uniform homeomorphism $h_g : K_F \to K_E$ with $h_g(0) = 0$ and a function $\tau_g \in C(K_F \setminus \{0\})$, with $|\tau_g(y)| = 1$ for all $y \in K_F \setminus \{0\}$, such that

$$Tg(y) = \begin{cases} \tau_g(y)g(h_g(y)), & y \in K_F \setminus \{0\}, \\ 0, & y = 0. \end{cases} \tag{2.2.8}$$

From Equations (2.2.7) and (2.2.8) we get $h = h_g$ on $K_F \setminus \{0\}$ implying that h is a uniform homeomorphism of $K_F \setminus \{0\}$ onto $K_E \setminus \{0\}$. The map h can be extended to a uniform homeomorphism of K_F onto K_E by defining $h(0) = 0$. This completes the proof. □

CHAPTER 3

Local isometries on $C^2[0,1]$

In this chapter, we investigate the structure of local isometries on $C^2[0,1]$, the Banach space of all functions that have continuous derivatives f', f'' on the closed unit interval $[0,1]$, equipped with norm $\|f\| = |f(0)| + |f'(0)| + \|f''\|_\infty$, where $\|\cdot\|_\infty$ is the usual supremum norm. We further characterize isometries of finite order, and study the structural properties of local isometries of finite order, on the space $C^2[0,1]$.

We recall that an isometry on a Banach space E is of finite order if there exists $n \in \mathbb{N}$ such that $T^n = I$, where I denotes the identity operator on E. For $n \in \mathbb{N}$, let $\mathcal{G}^n(E) = \{T \in \mathcal{G}(E) : T^n = I\}$.

The content of this chapter is entirely taken from [6].

Remark 3.0.1. *Let $T : C^2[0,1] \to C^2[0,1]$ be a local isometry, i.e., $T \in \overline{\mathcal{G}(C^2[0,1])}^a$.*

1. *For every $f \in C^2[0,1]$, there exists $T_f \in \mathcal{G}(C^2[0,1])$ such that $Tf = T_f f$. An application of Theorem 1.4.7 imply the existence of a homeomorphism $\phi_f : [0,1] \to [0,1]$, a continuous function $\omega_f : [0,1] \to \mathbb{T}$ and constants $\lambda_f, \mu_f \in \mathbb{T}$ such that T_f will be either an isometry of the first type (form $(\star)$) or an isometry of the second*

type ((⋆⋆)). That is,

$$Tf(x) = T_f f(x) = \lambda_f f(0) + \mu_f f'(0)x + (\zeta^2(\omega_f(f'' \circ \phi_f)))(x), \tag{3.0.1}$$

or

$$Tf(x) = T_f f(x) = \lambda_f f'(0) + \mu_f f(0)x + (\zeta^2(\omega_f(f'' \circ \phi_f)))(x). \tag{3.0.2}$$

2. *In this chapter, we will use the fact stated here in point (1) again and again. To avoid repetition, we will write the expressions of the local isometry given in equations* (3.0.1) *and* (3.0.2) *without mentioning the details of the maps ϕ_f, ω_f and the constants λ_f, μ_f. It will be understood from the subscript f in ϕ_f, ω_f and λ_f, μ_f that they are related to the function f and the isometry T_f.*

3. *We will also compute $Tf(x)$ and $(Tf)'(x)$ at $x = 0$ for various functions. In this case, Equations* (3.0.1) *and* (3.0.2) *imply that $Tf(0) = \lambda_f f(0)$, $(Tf)'(0) = \mu_f f'(0)$ or $Tf(0) = \lambda_f f'(0)$, $(Tf)'(0) = \mu_f f(0)$. Moreover, $(Tf)'' = (T_f f)'' = \omega_f(f'' \circ \phi_f)$.*

The following innocent remark is the key idea in our proof.

Remark 3.0.2. *For any $T \in B(C^2[0,1])$ and $g \in C^2[0,1]$, it is easy to verify that*

$$Tg(x) = Tg(0) + (Tg)'(0)x + (\zeta^2((Tg)''))(x). \tag{3.0.3}$$

3.1 Local isometries on $C^2[0,1]$

In this section, we state and prove the main result of this chapter. For the sake of clarity, the proof is divided into six steps.

Theorem 3.1.1. *$\mathcal{G}(C^2[0,1])$ is algebraically reflexive.*

Proof. Let $T \in \overline{\mathcal{G}(C^2[0,1])}^a$ and $g \in C^2[0,1]$. We use Remark 3.0.2 to show that T has either form ($\star$) or ($\star\star$) by finding expressions of $Tg(0)$, $(Tg)'(0)$ and $(Tg)''$ independent of g.

We complete the proof in several steps.

Step I. Define a linear map $W : C[0,1] \to C[0,1]$ by $W(f) = (T(\zeta^2 f))''$. We claim that W is a local isometry on $C[0,1]$. To see this, consider a function $f \in C[0,1]$. Then $\zeta^2 f = h$ (say) $\in C^2[0,1]$. Since T is local isometry, then there exists $T_h \in \mathcal{G}(C^2[0,1])$ such that $Th = T_h h$. This implies that $W(f) = (Th)'' = (T_h h)'' = \omega_h(h'' \circ \phi_h) = \omega_h(f \circ \phi_h)$ (see point (3) of Remark 3.0.1). Therefore, W is a local isometry. By [44, Theorem 2.2] W is a surjective linear isometry. Hence, there exist a continuous functions $\omega : [0,1] \to \mathbb{T}$ and a homeomorphism $\phi : [0,1] \to [0,1]$ such that $W(f) = \omega(f \circ \phi)$.

Step II. Let $k = \zeta^2 g''$. We observe that $k \in C^2[0,1]$ and $(Tg)'' - (Tk)'' = (Tg - Tk)'' = (T(g-k))''$. As T is a local isometry, there exists $T_{g-k} \in \mathcal{G}(C^2[0,1])$ such that $T(g-k) = T_{g-k}(g-k)$. This implies that

$$(T(g-k))'' = (T_{g-k}(g-k))'' = \omega_{g-k}((g-k)'' \circ \phi_{g-k}) = 0.$$

Hence,

$$(Tg)'' = (Tk)'' = (T(\zeta^2 g''))'' = W(g'') = \omega(g'' \circ \phi). \tag{3.1.1}$$

Step III. There exists $T_g \in \mathcal{G}(C^2[0,1])$ such that $Tg = T_g g$. Computing $Tg(x)$ and $(Tg)'(x)$ at $x = 0$, we get the following two cases (point (3) of Remark 3.0.1).

Case 1. $Tg(0) = \lambda_g g(0)$, $(Tg)'(0) = \mu_g g'(0)$.

Case 2. $Tg(0) = \lambda_g g'(0)$, $(Tg)'(0) = \mu_g g(0)$.

Step IV. For the functions $f = 1$, $f = id$, there exist T_1, T_{id} in $\mathcal{G}(C^2[0,1])$ respectively such that $T1 = T_1 1$, $Tid = T_{id} id$. So, we have the following four cases.

Case 3. $T1 = \lambda_1 1$, $Tid = \mu_{id} id$.

Case 4. $T1 = \lambda_1 1$, $Tid = \lambda_{id} 1$.

Case 5. $T1 = \mu_1 id$, $Tid = \mu_{id} id$.

Case 6. $T1 = \mu_1 id$, $Tid = \lambda_{id} 1$.

Cases 4 and **5** will lead to a contradiction. To see this, we consider **Case 4**. Let $T_{1+id} \in \mathcal{G}(C^2[0,1])$ such that $T(1+id) = T_{1+id}(1+id)$. Indeed, for every $x \in [0,1]$ we have,

$$\lambda_1 + \lambda_{id} = T(1)(x) + T(id)(x) = T(1+id)(x) = \lambda_{1+id} + \mu_{1+id} x,$$

a contradiction. Similarly, one can see that **Case 5** is also not possible.

Step V. Let $f_1 = g - g(0)1$ and $f_2 = g - g'(0)id$. There exist $T_{f_1}, T_{f_2} \in \mathcal{G}(C^2[0,1])$ such that $Tf_1 = T_{f_1}f_1$ and $Tf_2 = T_{f_2}f_2$. Now, using linearity of T we have the following equations.

$$Tf_1(0) = Tg(0) - g(0)T1(0), \ (Tf_1)'(0) = (Tg)'(0) - g(0)(T1)'(0) \tag{3.1.2}$$

and

$$Tf_2(0) = Tg(0) - g'(0)Tid(0), \ (Tf_2)'(0) = (Tg)'(0) - g'(0)(Tid)'(0). \tag{3.1.3}$$

Moreover, using Equations ($\star$) and ($\star\star$) and the local structure of T we get the following cases.

Case 7. $Tf_1(0) = 0$, $(Tf_1)'(0) = \mu_{f_1}g'(0)$, $Tf_2(0) = \lambda_{f_2}g(0)$, $(Tf_2)'(0) = 0$.

Case 8. $Tf_1(0) = 0$, $(Tf_1)'(0) = \mu_{f_1}g'(0)$, $Tf_2(0) = 0$, $(Tf_2)'(0) = \mu_{f_2}g(0)$.

Case 9. $Tf_1(0) = \lambda_{f_1}g'(0)$, $(Tf_1)'(0) = 0$, $Tf_2(0) = \lambda_{f_2}g(0)$, $(Tf_2)'(0) = 0$.

Case 10. $Tf_1(0) = \lambda_{f_1}g'(0)$, $(Tf_1)'(0) = 0$, $Tf_2(0) = 0$, $(Tf_2)'(0) = \mu_{f_2}g(0)$.

Step VI. In this step, using Equations (3.1.2) and (3.1.3) we consider the cases arose in steps **III, IV** and **V** one by one.

Cases 1 and 3. $Tg(0) = \lambda_g g(0)$, $(Tg)'(0) = \mu_g g'(0)$, $T1 = \lambda_1 1$ and $T(id) = \mu_{id} id$.

(i) If Case 7 holds, then Equations (3.1.2) and (3.1.3) imply that

$\lambda_g g(0) - g(0)\lambda_1 = 0$, $\mu_g g'(0) = \mu_{f_1}g'(0)$, $\lambda_g g(0) = \lambda_{f_2}g(0)$, $\mu_g g'(0) - g'(0)\mu_{id} = 0$.

Hence, $\lambda_g g(0) = \lambda_1 g(0)$ and $\mu_g g'(0) = \mu_{id}g'(0)$.

Putting the values of $\lambda_g g(0)$ and $\mu_g g'(0)$ obtained here, and the value of $(Tg)''$ from Equation (3.1.1) in Equation (3.0.3) we get

$$\begin{aligned} Tg(x) &= Tg(0) + (Tg)'(0)x + (\zeta^2((Tg)''))(x) \\ &= \lambda_1 g(0) + \mu_{id}g'(0)x + (\zeta^2(\omega(g'' \circ \phi)))(x). \end{aligned}$$

(ii) If Case 8 holds, then we have

$\lambda_g g(0) - g(0)\lambda_1 = 0$, $\mu_g g'(0) = \mu_{f_1} g'(0)$, $\lambda_g g(0) = 0$, $\mu_g g'(0) - g'(0)\mu_{id} = \mu_{f_2} g(0)$.

Thus, $g(0) = 0$ and $\mu_g g'(0) = \mu_{id} g'(0)$.

Equations (3.1.1) and (3.0.3) imply that

$$\begin{aligned} Tg(x) &= Tg(0) + (Tg)'(0)x + (\zeta^2((Tg)''))(x) \\ &= \mu_{id} g'(0)x + (\zeta^2(\omega(g'' \circ \phi)))(x). \end{aligned}$$

(iii) If Case 9 holds, then

$\lambda_g g(0) - g(0)\lambda_1 = \lambda_{f_1} g'(0)$, $\mu_g g'(0) = 0$, $\lambda_g g(0) = \lambda_{f_2} g(0)$, $\mu_g g'(0) - g'(0)\mu_{id} = 0$.

It follows that $g'(0) = 0$ and $\lambda_g g(0) = \lambda_1 g(0)$.

As a consequence of Equations (3.1.1) and (3.0.3) we have

$$\begin{aligned} Tg(x) &= Tg(0) + (Tg)'(0)x + (\zeta^2((Tg)''))(x) \\ &= \lambda_1 g(0) + (\zeta^2(\omega(g'' \circ \phi)))(x). \end{aligned}$$

(iv) If Case 10 holds, then

$\lambda_g g(0) - g(0)\lambda_1 = \lambda_{f_1} g'(0)$, $\mu_g g'(0) = 0$, $\lambda_g g(0) = 0$, $\mu_g g'(0) - g'(0)\mu_{id} = \mu_{f_2} g(0)$.

This implies that $g(0) = 0$ and $g'(0) = 0$.

Again, we apply Equations (3.1.1) and (3.0.3) to get

$$Tg(x) = Tg(0) + (Tg)'(0)x + (\zeta^2((Tg)''))(x) = (\zeta^2(\omega(g'' \circ \phi)))(x).$$

It follows from the expressions of T obtained in (i)-(iv) that $T \in \mathcal{G}(C^2[0,1])$.

Cases 1 and 6. $Tg(0) = \lambda_g g(0)$, $(Tg)'(0) = \mu_g g'(0)$, $T1 = \mu_1 id$ and $T(id) = \lambda_{id} 1$.

(i) If Case 7 holds, then

$\lambda_g g(0) = 0$, $\mu_g g'(0) - g(0)\mu_1 = \mu_{f_1} g'(0)$, $\lambda_g g(0) - g'(0)\lambda_{id} = \lambda_{f_2} g(0)$, $\mu_g g'(0) = 0$.

This implies that $g(0) = 0$ and $g'(0) = 0$.

(ii) If Case 8 holds, then we have

$$\lambda_g g(0) = 0,\ \mu_g g'(0) - g(0)\mu_1 = \mu_{f_1} g'(0),\ \lambda_g g(0) - g'(0)\lambda_{id} = 0,\ \mu_g g'(0) = \mu_{f_2} g(0).$$

Whence, $g(0) = 0$ and $g'(0) = 0$.

(iii) If Case 9 holds, then

$$\lambda_g g(0) = \lambda_{f_1} g'(0),\ \mu_g g'(0) - g(0)\mu_1 = 0,\ \lambda_g g(0) - g'(0)\lambda_{id} = \lambda_{f_2} g(0),\ \mu_g g'(0) = 0.$$

Thus, $g(0) = 0$ and $g'(0) = 0$.

(iv) If Case 10 holds, then

$$\lambda_g g(0) = \lambda_{f_1} g'(0),\ \mu_g g'(0) - g(0)\mu_1 = 0,\ \lambda_g g(0) - g'(0)\lambda_{id} = 0,\ \mu_g g'(0) = \mu_{f_2} g(0).$$

Consequently, $\lambda_g g(0) = \lambda_{id} g'(0)$, $\mu_g g'(0) = \mu_1 g(0)$.

Putting the values of $\lambda_g g(0)$ and $\mu_g g'(0)$ obtained in (i)-(iv), and the value of $(Tg)''$ from Equation (3.1.1), in Equation (3.0.3) we conclude that $T \in \mathcal{G}(C^2[0,1])$.

Cases 2 and 3. $Tg(0) = \lambda_g g'(0)$, $(Tg)'(0) = \mu_g g(0)$, $T1 = \lambda_1 1$ and $T(id) = \mu_{id} id$.

(i) If Case 7 holds, then

$$\lambda_g g'(0) - g(0)\lambda_1 = 0,\ \mu_g g(0) = \mu_{f_1} g'(0),\ \lambda_g g'(0) = \lambda_{f_2} g(0),\ \mu_g g(0) - g'(0)\mu_{id} = 0.$$

Hence, $\lambda_g g'(0) = \lambda_1 g(0)$, $\mu_g g(0) = \mu_{id} g'(0)$.

(ii) If Case 8 holds, then we have

$$\lambda_g g'(0) - g(0)\lambda_1 = 0,\ \mu_g g(0) = \mu_{f_1} g'(0),\ \lambda_g g'(0) = 0,\ \mu_g g(0) - g'(0)\mu_{id} = \mu_{f_2} g(0).$$

This implies that $g(0) = 0$ and $g'(0) = 0$.

(iii) If Case 9 holds, then

$$\lambda_g g'(0) - g(0)\lambda_1 = \lambda_{f_1} g'(0),\ \mu_g g(0) = 0,\ \lambda_g g'(0) = \lambda_{f_2} g(0),\ \mu_g g(0) - g'(0)\mu_{id} = 0.$$

Thus, $g(0) = 0$ and $g'(0) = 0$.

(iv) If Case 10 holds, then

$$\lambda_g g'(0) - g(0)\lambda_1 = \lambda_{f_1} g'(0),\ \mu_g g(0) = 0,\ \lambda_g g'(0) = 0,\ \mu_g g(0) - g'(0)\mu_{id} = \mu_{f_2} g(0).$$

It follows that $g(0) = 0$ and $g'(0) = 0$.

Proceeding exactly as above, we get $T \in \mathcal{G}(C^2[0,1])$.

Cases 2 and 6. $Tg(0) = \lambda_g g'(0)$, $(Tg)'(0) = \mu_g g(0)$, $T1 = \mu_1 id$ and $T(id) = \lambda_{id}1$.

(i) If Case 7 holds, then

$\lambda_g g'(0) = 0$, $\mu_g g(0) - g(0)\mu_1 = \mu_{f_1} g'(0)$, $\lambda_g g'(0) - g'(0)\lambda_{id} = \lambda_{f_2} g(0)$, $\mu_g g(0) = 0$.

Whence, $g(0) = 0$ and $g'(0) = 0$.

(ii) If Case 8 holds, then we have

$\lambda_g g'(0) = 0$, $\mu_g g(0) - g(0)\mu_1 = \mu_{f_1} g'(0)$, $\lambda_g g'(0) - g'(0)\lambda_{id} = 0$, $\mu_g g(0) = \mu_{f_2} g(0)$.

It follows that $g'(0) = 0$ and $\mu_g g(0) = \mu_1 g(0)$.

(iii) If Case 9 holds, then

$\lambda_g g'(0) = \lambda_{f_1} g'(0)$, $\mu_g g(0) - g(0)\mu_1 = 0$, $\lambda_g g'(0) - g'(0)\lambda_{id} = \lambda_{f_2} g(0)$, $\mu_g g(0) = 0$.

Thus, $g(0) = 0$ and $\lambda_g g'(0) = \lambda_{id} g'(0)$.

(iv) If Case 10 holds, then

$\lambda_g g'(0) = \lambda_{f_1} g'(0)$, $\mu_g g(0) - g(0)\mu_1 = 0$, $\lambda_g g'(0) - g'(0)\lambda_{id} = 0$, $\mu_g g(0) = \mu_{f_2} g(0)$.

As a consequence, we have $\mu_g g(0) = \mu_1 g(0)$, $\lambda_g g'(0) = \lambda_{id} g'(0)$.

Repeating the above process, we conclude that $T \in \mathcal{G}(C^2[0,1])$.

Thus, the proof is complete. □

3.2 Structure of isometries of finite order on $C^2[0,1]$

The structure of finite order isometries on $C^2[0,1]$ is given in the following proposition.

Proposition 3.2.1. *$T \in \mathcal{G}^n(C^2[0,1])$ if and only if there exist $\lambda, \mu \in \mathbb{T}$, a homeomorphism $\phi : [0,1] \to [0,1]$, and a continuous function $\omega : [0,1] \to \mathbb{T}$ such that for all $f \in C^2[0,1]$ and $x \in [0,1]$, one of the following cases occurs.*

1. *If T is an isometry of the first type, then*

$$\lambda^n = \mu^n = 1,\ \omega(x)\omega(\phi(x))\omega(\phi^2(x))\cdots\omega(\phi^{n-1}(x)) = 1,\ \phi^n(x) = x.$$

2. *If T is an isometry of the second type, then n is even and*

$$\lambda^{\frac{n}{2}}\mu^{\frac{n}{2}} = 1,\ \omega(x)\omega(\phi(x))\omega(\phi^2(x))\cdots\omega(\phi^{n-1}(x)) = 1,\ \phi^n(x) = x.$$

Proof. We first prove the *'only if'* part.

Let $T \in \mathcal{G}^n(C^2[0,1])$. We first note that since $T \in \mathcal{G}(C^2[0,1])$, $\exists\ \lambda, \mu \in \mathbb{T}$, a homeomorphism $\phi : [0,1] \to [0,1]$ and a continuous function $\omega : [0,1] \to \mathbb{T}$ such that T is of either form $(\star)$ or $(\star\star)$.

Suppose T has the form $(\star)$, then $T^n = I$ implies that

$$\begin{aligned}&\lambda^n f(0) + \mu^n f'(0)x + \\ &\qquad \int_0^x \int_0^t \omega(s)\omega(\phi(s))\omega(\phi^2(s))\cdots\omega(\phi^{n-1}(s)) f''(\phi^n(s))\,ds\,dt = f(x).\end{aligned} \tag{3.2.1}$$

If we put $f = 1$ in the above equation we get $\lambda^n = 1$. If we differentiate the above equation and put $f = id$ we get $\mu^n = 1$. Taking the second derivative of Equation (3.2.1) we have

$$\omega(x)\omega(\phi(x))\omega(\phi^2(x))\cdots\omega(\phi^{n-1}(x)) f''(\phi^n(x)) = f''(x).$$

This implies that

$$\omega(x)\omega(\phi(x))\omega(\phi^2(x))\cdots\omega(\phi^{n-1}(x)) = 1,\ \text{and}\ \phi^n(x) = x.$$

Now, suppose T has the form $(\star\star)$. We consider the following two cases.

If n is odd, then $T^n = I$ will implies that

$$\begin{aligned}&\lambda^{\frac{n+1}{2}}\mu^{\frac{n-1}{2}} f'(0) + \lambda^{\frac{n-1}{2}}\mu^{\frac{n+1}{2}} f(0)x + \\ &\qquad \int_0^x \int_0^t \omega(s)\omega(\phi(s))\omega(\phi^2(s))\cdots\omega(\phi^{n-1}(s)) f''(\phi^n(s))\,ds\,dt = f(x).\end{aligned}$$

Considering $f = id$ in the above equation, we get $\lambda^{\frac{n+1}{2}}\mu^{\frac{n-1}{2}} = x$, for all $x \in [0,1]$, a contradiction.

If n is even then

$$\lambda^{\frac{n}{2}}\mu^{\frac{n}{2}}f(0)+\lambda^{\frac{n}{2}}\mu^{\frac{n}{2}}f'(0)x+ \int_0^x\int_0^t \omega(s)\omega(\phi(s))\omega(\phi^2(s))\cdots\omega(\phi^{n-1}(s))f''(\phi^n(s))dsdt = f(x).$$

Putting $f=1$ in the above equation we have $\lambda^{\frac{n}{2}}\mu^{\frac{n}{2}}=1$. If we differentiate the above equation twice we have

$$\omega(x)\omega(\phi(x))\omega(\phi^2(x))\cdots\omega(\phi^{n-1}(x))f''(\phi^n(x)) = f''(x).$$

It follows that

$$\omega(x)\omega(\phi(x))\omega(\phi^2(x))\cdots\omega(\phi^{n-1}(x)) = 1, \text{ and } \phi^n(x)=x.$$

For the *'if'* part, suppose that assertion (1) holds. Then

$$\begin{aligned}T^nf(x) &= \lambda^nf(0)+\mu^nf'(0)x+\int_0^x\int_0^t \omega(s)\omega(\phi(s))\omega(\phi^2(s))\cdots\omega(\phi^{n-1}(s))f''(\phi^n(s))dsdt\\ &= f(0)+f'(0)+f(x)-f(0)-f'(0)\\ &= f(x).\end{aligned}$$

If assertion (2) holds, then we have

$$\begin{aligned}T^nf(x) &= \lambda^{\frac{n}{2}}\mu^{\frac{n}{2}}f(0)+\lambda^{\frac{n}{2}}\mu^{\frac{n}{2}}f'(0)x+\int_0^x\int_0^t \omega(s)\omega(\phi(s))\omega(\phi^2(s))\cdots\omega(\phi^{n-1}(s))f''(\phi^n(s))dsdt\\ &= f(0)+f'(0)+f(x)-f(0)-f'(0)\\ &= f(x).\end{aligned}$$

□

3.3 Local isometries of finite order on $C^2[0,1]$

We prove that the set $\mathcal{G}^n(C^2([0,1])$ is algebraically reflexive when n is odd. The case when n is even is discussed at the end of this section in the form of a remark.

Proposition 3.3.1. *If n is odd, then $\mathcal{G}^n(C^2[0,1])$ is algebraically reflexive.*

Proof. Let $T \in \overline{\mathcal{G}^n(C^2[0,1])}^a$. Then by using Theorem 3.1.1, we have $T \in \mathcal{G}(C^2[0,1])$. Theorem 1.4.7 imply the existence of $\lambda, \mu \in \mathbb{T}$, a homeomorphism $\phi : [0,1] \to [0,1]$ and a continuous map $\omega : [0,1] \to \mathbb{T}$ such that T is either of form $(\star)$ or $(\star\star)$. Moreover, for each $f \in C^2[0,1]$, there exists $T_f \in \mathcal{G}^n(C^2[0,1])$ such that $Tf = T_f f$. Since n is odd, we note that T_f will always be of form $(\star)$. Furthermore, $(Tf)''(x) = (T_f f)''(x)$ implies that $\omega(x)f''(\phi(x)) = \omega_f(x)f''(\phi_f(x))$. Taking $f = x^3$ we get $\omega(x) = \omega_f(x)$ and $\phi(x) = \phi_f(x)$ for every $x \in [0,1]$. This implies that

$$\omega(x)\omega(\phi(x))\omega(\phi^2(x))\cdots\omega(\phi^{n-1}(x)) = 1 \text{ and } \phi^n(x) = x.$$

Now, suppose that T is of form $(\star)$. Then computing Tf and $(Tf)'$ at $x = 0$ we get $\lambda f(0) = \lambda_f f(0)$ and $\mu f'(0) = \mu_f f'(0)$. This implies that $\lambda = \lambda_f$ and $\mu = \mu_f$. Therefore, $\lambda^n = \mu^n = 1$. Hence, $T \in \mathcal{G}^n(C^2[0,1])$.

If T is of form $(\star\star)$, then repeating the same calculations yield $\lambda f'(0) = \lambda_f f(0)$ and $\mu f(0) = \mu_f f'(0)$. Choosing $f \in C^2[0,1]$ such that $f(0) = 0$, $f'(0) \neq 0$ will lead to a contradiction. This completes the proof. $\square$

We complete this chapter with the following remark.

Remark 3.3.2. *The problem of algebraic reflexivity of $\mathcal{G}^n(C^2[0,1])$ remains open when n is even. Let $T \in \overline{\mathcal{G}^n(C^2[0,1])}^a$ and $f \in C^2[0,1]$ such that $f(0) = f'(0) \neq 0$. If T and T_f are of the same form, then proceeding as we did in Proposition 3.3.1 we can show that $T \in \mathcal{G}^n(C^2[0,1])$. We consider two cases.*

1. *If T is of form $(\star)$ and T_f is of form $(\star\star)$, then $\lambda f(0) = \lambda_f f'(0)$ and $\mu f'(0) = \mu_f f(0)$. Whence $\lambda = \lambda_f$ and $\mu = \mu_f$. We also have $\lambda_f^{n/2}\mu_f^{n/2} = 1$ but $\lambda^n = \mu^n = 1$ may not be true. For example, let $n = 2$, and $\lambda = i$, $\mu = -i$.*

2. *If T is of form $(\star\star)$ and T_f is of form $(\star)$, then $\lambda f'(0) = \lambda_f f(0)$ and $\mu f(0) = \mu_f f'(0)$. This implies that $\lambda = \lambda_f$ and $\mu = \mu_f$. Moreover, $\lambda_f^n = \mu_f^n = 1$. In this case, $\lambda^{n/2}\mu^{n/2} = 1$ may not always be true. For example, if $n = 2$, and $\lambda = 1$, $\mu = -1$.*

CHAPTER 4

Projections in the convex hull of isometries on $C^2[0,1]$

In this chapter, we characterize projections on $C^2[0,1]$ that can be written as a convex combination of two surjective linear isometries. We also find out the structure of Hermitian projections and generalized bi-circular projections on $C^2[0,1]$. Finally, we discuss the relationship of these two types of projections (Hermitian and generalized bi-circular projections) with the convex combination of two isometries.

All the results of this chapter have appeared in [7].

4.1 Projections as a convex combination of two surjective isometries on $C^2[0,1]$

To investigate when a convex combination of isometries on a Banach space is a projection, we observe that it is enough to consider the average of these isometries as shown in the next lemma.

Lemma 4.1.1. *Let T_1 and T_2 be linear isometries (not necessarily surjective) on a Banach* *space X, and $P = \alpha_1 T_1 + \alpha_2 T_2$; where $\alpha_1, \alpha_2 > 0$, and $\alpha_1 + \alpha_2 = 1$. If P is a proper projection, then $\alpha_1 = \alpha_2 = \frac{1}{2}$.*

Proof. Since P is proper, there exists $x \neq 0$ such that $Px = 0$. Thus, $\alpha_1 T_1 x = -\alpha_2 T_2 x$. Since T_1, T_2 are isometries, taking norms on both sides we get $\alpha_1 = \alpha_2 = \frac{1}{2}$. □

Some notations which will be used in this chapter are given below in the form of a remark.

Remark 4.1.2. *1. An isometry of the first type (form $(\star)$) will be denoted by F. If we consider two isometries of the first type, we will denote them by F_1 and F_2.*

2. An isometry of the second type (form $(\star\star)$) will be denoted by S. Two isometries of second the type will be denote by S_1 and S_2.

3. The scalars λ, μ and the maps ϕ, ω associated with F_1 and F_2 (S_1 and S_2, F and S or a general isometry T_1 and T_2) will be denoted by λ_1, μ_1, ϕ_1, ω_1 and λ_2, μ_2, ϕ_2, ω_2, respectively. To avoid any repetition, we will not mention the details of these scalars and maps in the sequel. They are given in Theorem 1.4.7.

Theorem 4.1.3. *Let T_1 and T_2 be surjective linear isometries on $C^2[0,1]$. Then $P = \frac{1}{2}(T_1 + T_2)$ is a projection if and only if the following assertions holds:*

1. The scalars λ_i, μ_i, $i = 1, 2$, satisfy one of the following conditions

a) $\lambda_1 = -\lambda_2$ or $\lambda_1 = \lambda_2 = 1$; $\mu_1 = -\mu_2$ or $\mu_1 = \mu_2 = 1$.

b) $\lambda_1 = \mu_1 = 1$ and $\lambda_2 \mu_2 = 1$;

2. Every $x \in [0,1]$ satisfies one of the following statements

a) $x = \phi_1(x) = \phi_2(x)$ and $\omega_1(x) = \omega_2(x) = 1$.

b) $\phi_1(x) = \phi_2(x)$ and $\omega_1(x) = -\omega_2(x)$.

c) For $i, j = 1, 2$, $i \neq j$, $x = \phi_i(x) \neq \phi_j(x)$, $\phi_j^2(x) = x$, $\phi_i \circ \phi_j(x) = \phi_j(x)$, $\omega_i(x) = \omega_i(\phi_j(x)) = 1$ and $\omega_j(x)\omega_j(\phi_j(x)) = 1$.

The proof of the theorem follows from the next four lemmas.

Lemma 4.1.4. *Let P be the projection on $C^2[0,1]$ such that $P = \frac{1}{2}(F_1 + F_2)$, where $F_1, F_2 \in \mathcal{G}(C^2[0,1])$. Then $\lambda_1 + \lambda_2 = 0$ or $\lambda_1 = \lambda_2 = 1$; $\mu_1 + \mu_2 = 0$ or $\mu_1 = \mu_2 = 1$.*

Proof. Since P is a projection we have

$$\begin{aligned}
&\frac{1}{2}(F_1 f(x) + F_2 f(x)) \\
&= \frac{1}{4}(F_1(F_1 f)(x) + F_1(F_2 f)(x) + F_2(F_1 f)(x) + F_2(F_2 f)(x)) \\
&= \frac{1}{4}\bigg(\lambda_1 (F_1 f)(0) + \mu_1 (F_1 f)'(0)x + \int_0^x \int_0^t \omega_1(s)(F_1 f)''(\phi_1(s))dsdt \\
&\qquad + \lambda_1 (F_2 f)(0) + \mu_1 (F_2 f)'(0)x + \int_0^x \int_0^t \omega_1(s)(F_2 f)''(\phi_1(s))dsdt \\
&\qquad + \lambda_2 (F_1 f)(0) + \mu_2 (F_1 f)'(0)x + \int_0^x \int_0^t \omega_2(s)(F_1 f)''(\phi_2(s))dsdt \\
&\qquad + \lambda_2 (F_2 f)(0) + \mu_2 (F_2 f)'(0)x + \int_0^x \int_0^t \omega_2(s)(F_2 f)''(\phi_2(s))dsdt\bigg) \\
&= \frac{1}{4}\bigg(\lambda_1^2 f(0) + \mu_1^2 f'(0)x + \int_0^x \int_0^t \omega_1(s)\omega_1(\phi_1(s))f''(\phi_1^2(s))dsdt \\
&\qquad + \lambda_1\lambda_2 f(0) + \mu_1\mu_2 f'(0)x + \int_0^x \int_0^t \omega_1(s)\omega_2(\phi_1(s))f''(\phi_2(\phi_1(s)))dsdt \\
&\qquad + \lambda_2\lambda_1 f(0) + \mu_2\mu_1 f'(0)x + \int_0^x \int_0^t \omega_2(s)\omega_1(\phi_2(s))f''(\phi_1(\phi_2(s)))dsdt \\
&\qquad + \lambda_2^2 f(0) + \mu_2^2 f'(0)x + \int_0^x \int_0^t \omega_2(s)\omega_2(\phi_2(s))f''(\phi_2^2(s))dsdt\bigg).
\end{aligned}$$

This implies that

$$\begin{aligned}
&2\bigg(\lambda_1 f(0) + \mu_1 f'(0)x + \int_0^x \int_0^t \omega_1(s)f''(\phi_1(s))dsdt \\
&+ \lambda_2 f(0) + \mu_2 f'(0)x + \int_0^x \int_0^t \omega_2(s)f''(\phi_2(s))dsdt\bigg) \\
&= \lambda_1^2 f(0) + \mu_1^2 f'(0)x + \int_0^x \int_0^t \omega_1(s)\omega_1(\phi_1(s))f''(\phi_1^2(s))dsdt \\
&\quad + \lambda_1\lambda_2 f(0) + \mu_1\mu_2 f'(0)x + \int_0^x \int_0^t \omega_1(s)\omega_2(\phi_1(s))f''(\phi_2(\phi_1(s)))dsdt \\
&\quad + \lambda_2\lambda_1 f(0) + \mu_2\mu_1 f'(0)x + \int_0^x \int_0^t \omega_2(s)\omega_1(\phi_2(s))f''(\phi_1(\phi_2(s)))dsdt
\end{aligned}$$

$$+ \lambda_2^2 f(0) + \mu_2^2 f'(0)x + \int_0^x \int_0^t \omega_2(s)\omega_2(\phi_2(s))f''(\phi_2^2(s))dsdt. \tag{4.1.1}$$

Now, if we consider the function $f = 1$, Equation (4.1.1) reduces to

$2(\lambda_1 + \lambda_2) = \lambda_1^2 + 2\lambda_1\lambda_2 + \lambda_2^2$. Thus, $\lambda_1 + \lambda_2 = 0$ or $\lambda_1 + \lambda_2 = 2$. Hence, $\lambda_1 = -\lambda_2$ or $\lambda_1 = \lambda_2 = 1$.

Similarly, when $f = id$, we have $2(\mu_1 + \mu_2)x = (\mu_1^2 + 2\mu_1\mu_2 + \mu_2^2)x$. Hence, $\mu_1 + \mu_2 = 0$ or $\mu_1 + \mu_2 = 2$. This implies that $\mu_1 = -\mu_2$ or $\mu_1 = \mu_2 = 1$. □

Lemma 4.1.5. *Let P be the projection on $C^2[0,1]$ such that $P = \frac{1}{2}(F + S)$, where $F, S \in \mathcal{G}(C^2[0,1])$. Then $\lambda_1 = \mu_1 = 1$ and $\lambda_2\mu_2 = 1$.*

Proof. $P = P^2$ implies that

$$\begin{aligned}
&\frac{1}{2}(Ff(x) + Sf(x)) \\
&= \frac{1}{2}\bigg(\lambda_1 f(0) + \mu_1 f'(0)x + \int_0^x \int_0^t \omega_1(s)f''(\phi_1(s))dsdt \\
&+ \lambda_2 f'(0) + \mu_2 f(0)x + \int_0^x \int_0^t \omega_2(s)f''(\phi_2(s))dsdt\bigg) \\
&= \frac{1}{4}(F(Ff)(x) + F(Sf)(x) + S(Ff)(x) + S(Sf)(x)) \\
&= \frac{1}{4}\bigg(\lambda_1(Ff)(0) + \mu_1(Ff)'(0)x + \int_0^x \int_0^t \omega_1(s)(Ff)''(\phi_1(s))dsdt \\
&\quad + \lambda_1(Sf)(0) + \mu_1(Sf)'(0)x + \int_0^x \int_0^t \omega_1(s)(Sf)''(\phi_1(s))dsdt \\
&\quad + \lambda_2(Ff)'(0) + \mu_2(Ff)(0)x + \int_0^x \int_0^t \omega_2(s)(Ff)''(\phi_2(s))dsdt \\
&\quad + \lambda_2(Sf)'(0) + \mu_2(Sf)(0)x + \int_0^x \int_0^t \omega_2(s)(Sf)''(\phi_2(s))dsdt\bigg) \\
&= \frac{1}{4}\bigg(\lambda_1^2 f(0) + \mu_1^2 f'(0)x + \int_0^x \int_0^t \omega_1(s)\omega_1(\phi_1(s))f''(\phi_1^2(s))dsdt \\
&\quad + \lambda_1\lambda_2 f'(0) + \mu_1\mu_2 f(0)x + \int_0^x \int_0^t \omega_1(s)\omega_2(\phi_1(s))f''(\phi_2(\phi_1(s)))dsdt \\
&\quad + \lambda_2\mu_1 f'(0) + \mu_2\lambda_1 f(0)x + \int_0^x \int_0^t \omega_2(s)\omega_1(\phi_2(s))f''(\phi_1(\phi_2(s)))dsdt \\
&\quad + \lambda_2\mu_2 f(0) + \mu_2\lambda_2 f'(0)x + \int_0^x \int_0^t \omega_2(s)\omega_2(\phi_2(s))f''(\phi_2^2(s)))dsdt\bigg).
\end{aligned} \tag{4.1.2}$$

Putting $f = 1$ in Equation (4.1.2), we get $2(\lambda_1+\mu_2 x) = \lambda_1^2+\mu_1\mu_2 x+\mu_2\lambda_1 x+\lambda_2\mu_2$. Solving this equation for $x = 0, 1$ we conclude $\lambda_1 = \mu_1 = 1$ and $\lambda_2\mu_2 = 1$. □

Lemma 4.1.6. *Let P be the projection on $C^2[0,1]$ such that $P = \frac{1}{2}(S_1 + S_2)$, where $S_1, S_2 \in \mathcal{G}(C^2[0,1])$. Then $\lambda_1 + \lambda_2 = 0$ and $\mu_1 + \mu_2 = 0$.*

Proof. As P is a projection, we have

$$\begin{aligned}
&\frac{1}{2}(S_1 f(x) + S_2 f(x))\\
&= \frac{1}{2}\Big(\lambda_1 f'0) + \mu_1 f(0)x + \int_0^x\int_0^t \omega_1(s) f''(\phi_1(s))dsdt\\
&+ \lambda_2 f'(0) + \mu_2 f(0)x + \int_0^x\int_0^t \omega_2(s) f''(\phi_2(s))dsdt\Big)\\
&= \frac{1}{4}(S_1(S_1 f)(x) + S_1(S_2 f)(x) + S_2(S_1 f)(x) + S_2(S_2 f)(x))\\
&= \frac{1}{4}\Big(\lambda_1(S_1 f)'(0) + \mu_1(S_1 f)(0)x + \int_0^x\int_0^t \omega_1(s)(S_1 f)''(\phi_1(s))dsdt\\
&\qquad + \lambda_1(S_2 f)'(0) + \mu_1(S_2 f)(0)x + \int_0^x\int_0^t \omega_1(s)(S_2 f)''(\phi_1(s))dsdt\\
&\qquad + \lambda_2(S_1 f)'(0) + \mu_2(S_1 f)(0)x + \int_0^x\int_0^t \omega_2(s)(S_1 f)''\phi_2(s))dsdt\\
&\qquad + \lambda_2(S_2 f)'(0) + \mu_2(S_2 f)(0)x + \int_0^x\int_0^t \omega_2(s)(S_2 f)''\phi_2(s))dsdt\Big)\\
&= \frac{1}{4}\Big(\lambda_1\mu_1 f(0) + \lambda_1\mu_1 f'(0)x + \int_0^x\int_0^t \omega_1(s)\omega_1(\phi_1(s))f''(\phi_1^2(s))dsdt\\
&\qquad + \lambda_1\mu_2 f(0) + \lambda_2\mu_1 f'(0)x + \int_0^x\int_0^t \omega_1(s)\omega_2(\phi_1(s))f''(\phi_2(\phi_1(s)))dsdt\\
&\qquad + \lambda_2\mu_1 f(0) + \mu_2\lambda_1 f'(0)x + \int_0^x\int_0^t \omega_2(s)\omega_1(\phi_2(s))f''(\phi_1(\phi_2(s)))dsdt\\
&\qquad + \lambda_2\mu_2 f(0) + \mu_2\lambda_2 f'(0)x + \int_0^x\int_0^t \omega_2(s)\omega_2(\phi_2(s))f''(\phi_2^2(s)))dsdt\Big).
\end{aligned} \tag{4.1.3}$$

For $f = 1$, Equation (4.1.3) becomes $2(\mu_1 + \mu_2)x = \lambda_1\mu_1 + \lambda_1\mu_2 + \lambda_2\mu_1 + \lambda_2\mu_2$. Whence, $\mu_1 + \mu_2 = 0$ or $\lambda_1 + \lambda_2 = 2x$. It follows that $\mu_1 = -\mu_2$.

If $f = id$ we get $2(\lambda_1 + \lambda_2) = (\lambda_1\mu_1 + \lambda_2\mu_1 + \mu_2\lambda_1 + \mu_2\lambda_2)x$. Hence, $\lambda_1 + \lambda_2 = 0$ or $(\mu_1 + \mu_2)x = 2$. Therefore, $\lambda_1 = -\lambda_2$. □

Lemma 4.1.7. *Let T_1 and T_2 be surjective linear isometries on $C^2[0,1]$. If $P = \frac{1}{2}(T_1+T_2)$ is a projection, then every $x \in [0,1]$ satisfies one of the following conditions.*

1. *$x = \phi_1(x) = \phi_2(x)$ and $\omega_1(x) = \omega_2(x) = 1$.*

2. *$\phi_1(x) = \phi_2(x)$ and $\omega_1(x) = -\omega_2(x)$.*

3. *For $i, j = 1, 2$, $i \neq j$, $x = \phi_i(x) \neq \phi_j(x)$, $\phi_j^2(x) = x$, $\phi_i \circ \phi_j(x) = \phi_j(x)$, $\omega_i(x) = \omega_i(\phi_j(x)) = 1$ and $\omega_j(x)\omega_j(\phi_j(x)) = 1$.*

Proof. Differentiating twice Equations (4.1.1), (4.1.2) and (4.1.3) we get

$$\begin{aligned} &2\Big(\omega_1(x)f''(\phi_1(x)) + \omega_2(x)f''(\phi_2(x))\Big) \\ &= \omega_1(x)\omega_1(\phi_1(x))f''(\phi_1^2(x)) + \omega_1(x)\omega_2(\phi_1(x))f''(\phi_2(\phi_1(x))) \\ &+ \omega_2(x)\omega_1(\phi_2(x))f''(\phi_1(\phi_2(x))) + \omega_2(x)\omega_2(\phi_2(x))f''(\phi_2^2(x)) \end{aligned} \tag{4.1.4}$$

for all $f \in C^2[0,1]$ and $x \in [0,1]$. We consider a partition of the unit interval $[0,1]$ as follows:

$A_1 = \{x \in [0,1] : \phi_1(x) = \phi_2(x) = x\}$, $A_2 = \{x \in [0,1] : \phi_1(x) = x \neq \phi_2(x)\}$,
$A_3 = \{x \in [0,1] : \phi_1(x) \neq \phi_2(x) = x\}$ and $A_4 = \{x \in [0,1] : \phi_1(x) \neq x,\ \phi_2(x) \neq x\}$.

Let $x \in A_1$. Equation (4.1.4) gives $2(\omega_1(x)+\omega_2(x)) = 4\omega_1(x)\omega_2(x)$ or $\omega_1(x) = \omega_2(x) = 1$.

Consider $x \in A_2$. We observe that $x \neq \phi_1 \circ \phi_2(x)$. We choose a function $f \in C^2[0,1]$ such that $f''(x) = 1$ and $f''(\phi_2(x)) = f''(\phi_1 \circ \phi_2(x)) = 0$, Equation (4.1.4) then reduces to

$$2\omega_1(x) = \omega_1(x)\omega_1(x) + \omega_2(x)\omega_2(\phi_2(x))f''(\phi_2^2(x)).$$

If $\phi_2^2(x) = x$, then $\omega_1(x) = 1$ and $\omega_2(x)\omega_2(\phi_2(x)) = 1$. If $\phi_2^2(x) \neq x$, then we get $2\omega_1(x) = \omega_1(x)\omega_1(x)$. This is a contradiction.

On the other hand, choose a function $f \in C^2[0,1]$ such that $f''(\phi_2(x)) = 1$ and $f''(x) = 0$. As a consequence, Equation (4.1.4) becomes $2\omega_2(x) = \omega_2(x)+\omega_2(x)\omega_1(\phi_2(x))f''(\phi_1(\phi_2(x)))$. Consequently, $\phi_1 \circ \phi_2(x) = \phi_2(x)$ and $\omega_1(\phi_2(x)) = 1$.

The case when $x \in A_3$ is similar.

Now, consider $x \in A_4$. If $\phi_1(x) = \phi_2(x)$, then we choose a function such that $f''(\phi_1(x)) = 1$ and $f''(\phi_1^2(x)) = f''(\phi_2^2(x)) = 0$. Hence, $\omega_1(x) = -\omega_2(x)$.

If $\phi_1(x) \neq \phi_2(x)$, then by taking an f such that $f''(\phi_1(x)) = 1$ and $f''(\phi_2(x)) = f''(\phi_1^2(x)) = f''(\phi_1 \circ \phi_2(x)) = 0$ in Equation (4.1.4) we get

$$2\omega_1(x) = \omega_1(x)\omega_2(\phi_1(x))f''(\phi_2(\phi_1(x))) + \omega_2(x)\omega_2(\phi_2(x))f''(\phi_2^2(x)).$$

This implies that $\phi_1(x)$ must be equal to exactly one of the points $\phi_2 \circ \phi_1(x)$ and $\phi_2^2(x)$. In either case, we have that $2\omega_1(x) = \omega_1(x)\omega_2(\phi_1(x))$ or $2\omega_1(x) = \omega_2(x)\omega_2(\phi_2(x))$. This is impossible. Thus, the proof is complete. $\square$

Completion of the proof of Theorem 4.1.3. The proof of part (1) of Theorem 4.1.3 follows from Lemmas 4.1.4, 4.1.5 and 4.1.6. Part (2) follows from Lemma 4.1.7. For the converse part, suppose assertion (2) holds. Then one can easily prove the following. If assertion (1)(a) holds, then $P = \frac{1}{2}(F_1 + F_2)$ is a projection. If assertion (1)(b) holds, then $P = \frac{1}{2}(F + S)$ is a projection. If $\lambda_1 + \lambda_2 = 0$ and $\mu_1 + \mu_2 = 0$, then $P = \frac{1}{2}(S_1 + S_2)$ is a projection. $\square$

4.2 Generalized bi-circular and Hermitian projections on $C^2[0,1]$

In this section, we give complete description of GBPs and Hermitian projections on $C^2[0,1]$. The main key to a possible characterization of GBPs on a Banach space is the following simple lemma.

Lemma 4.2.1. *Let X be a Banach space and $\alpha \in \mathbb{T} \setminus \{1\}$. Then the following are equivalent.*

1. *$T \in B(X)$ such that $T^2 - (\alpha + 1)T + \alpha I = 0$.*

2. *There exists a projection P on X such that $P + \alpha(I - P) = T$.*

Proof. (1) $\Longrightarrow$ (2) We define $P = \frac{T-\alpha I}{1-\alpha}$. Then we have $P + \alpha(I-P) = T$ and

$$\begin{aligned} P^2 &= \frac{T^2+\alpha^2-2\alpha T}{(1-\alpha)^2} \\ &= \frac{(\alpha+1)T-\alpha I+\alpha^2-2\alpha T}{(1-\alpha)^2} \\ &= \frac{(1-\alpha)(T-\alpha I)}{(1-\alpha)^2} \\ &= P. \end{aligned}$$

(2) $\Longrightarrow$ (1)

$$\begin{aligned} T^2-(\alpha+1)T+\alpha I &= P+\alpha^2(I-P)-(\alpha+1)[P+\alpha(I-P)]+\alpha I \\ &= -\alpha P+[\alpha^2-\alpha(\alpha+1)](I-P)+\alpha I \\ &= 0. \end{aligned}$$

□

Next, we state a lemma which describes the relation between Hermitian projections and bi-circular projections.

Lemma 4.2.2. *[36, Lemma 2.1] A projection on a Banach space is bi-circular if and only if it is a Hermitian projection.*

Theorem 4.2.3. *An operator P on $C^2[0,1]$ is a generalized bi-circular projection if and only if one of the following assertions holds:*

1. *If the isometry T associated with P is of the first type, then either*

   a) $P = \frac{I+T}{2}$, $\lambda = \pm 1$, $\mu = \pm 1$, and for every $x \in [0,1]$, $\omega(x)\omega(\phi(x)) = 1$, $\phi^2(x) = x$, or

   b) P is Hermitian. In this case, $\lambda = \alpha$ or 1, $\mu = \alpha$ or 1.

2. *If the isometry T associated with P is of the second type, then $P = \frac{I+T}{2}$, $\lambda\mu = 1$, and for every $x \in [0,1]$, $\omega(x)\omega(\phi(x)) = 1$, $\phi^2(x) = x$.*

Proof. The converse part is clear. Indeed, if assertion $1(a)$ is holds then $P-(I-P)=T$. This implies that P is a GBP.

If assertion $1(b)$ holds, then P is again a GBP by Lemma 4.2.2.

If case (2) occurs, then $P-(I-P)=T$, which implies that P is a GBP.

For the direct part, let $P+\alpha(I-P)=T$ for some $T\in\mathcal{G}(C^2[0,1])$ and $\alpha\in\mathbb{T}\setminus\{1\}$. By Lemma 4.2.1,

$$T^2f(x)-(1+\alpha)Tf(x)+\alpha f(x)=0.$$

We consider two cases.

Case I. Suppose T is of form $(\star)$, then

$$\begin{aligned}&\lambda^2 f(0)+\mu^2 f'(0)x+\int_0^x\int_0^t \omega(s)\omega(\phi(s))f''(\phi^2(s))dsdt\\ &-(1+\alpha)\left(\lambda f(0)+\mu f'(0)x+\int_0^x\int_0^t \omega(s)f''(\phi(s))\,ds\,dt\right)+\alpha f(x)=0.\end{aligned}\tag{4.2.1}$$

For $f=1$ Equation (4.2.1) becomes $\lambda^2-(1+\alpha)\lambda+\alpha=0$. Thus, $\lambda=\alpha$ or 1.

Taking $f=id$ we have $\mu^2x-(1+\alpha)\mu x+\alpha x=0$ which implies that $\mu=\alpha$ or 1.

Differentiating Equation (4.2.1) two times

$$\omega(x)\omega(\phi(x))f''(\phi^2(x))-(1+\alpha)\omega(x)f''(\phi(x))+\alpha f''(x)=0.\tag{4.2.2}$$

Let $A=\{x\in[0,1]:x\neq\phi(x)\}$. Assume $A\neq\emptyset$ and $x\in A$. If $x,\phi(x)$ and $\phi^2(x)$ are all distinct, then we can choose a function $f\in C^2[0,1]$ such that $f''(x)=1$ and $f''(\phi(x))=f''(\phi^2(x))=0$. So, Equation (4.2.2) gives $\alpha=0$ which is a contradiction. If $x=\phi^2(x)$, choose $f\in C^2[0,1]$ such that $f''(x)=1$ and $f''(\phi(x))=0$. Whence, Equation (4.2.2) reduces to $\omega(x)\omega(\phi(x))+\alpha=0$. On the other hand for $f(x)=x^2$, we have $\omega(x)\omega(\phi(x))-(1+\alpha)\omega(x)+\alpha=0$. Consequently, $\alpha=-1$ and $\omega(x)\omega(\phi(x))=1$.

On the other hand, if $A=\emptyset$, then Equation (4.2.2) reduces to $\omega^2(x)-(1+\alpha)\omega(x)+\alpha=0$. This implies that $\omega(x)=1$ or α. Since the set $[0,1]$ is connected, ω is constant. Suppose $\omega(x)=1$ for all $x\in[0,1]$. Then for any $f\in C^2[0,1]$ and $x\in[0,1]$

$$Pf(x)=\frac{(1-\alpha)f(x)+(\lambda-1)f(0)+(\mu-1)f'(0)x}{1-\alpha}.$$

If $\lambda = \mu = \alpha$, $Pf(x) = f(x) - f(0) - f'(0)x$.

If $\lambda = \alpha$, $\mu = 1$, $Pf(x) = f(x) - f(0)$.

If $\lambda = 1$, $\mu = \alpha$, $Pf(x) = f(x) - f'(0)x$.

If $\lambda = \mu = 1$, $Pf(x) = f(x)$.

Now, if $\omega(x) = \alpha$ for all $x \in [0,1]$. Then

$$Pf(x) = \frac{(\lambda - \alpha)f(0) + (\mu - \alpha)f'(0)x}{1 - \alpha}.$$

If $\lambda = \mu = \alpha$, $Pf(x) = 0$.

If $\lambda = \alpha$, $\mu = 1$, $Pf(x) = f'(0)x$.

If $\lambda = 1$, $\mu = \alpha$, $Pf(x) = f(0)$.

If $\lambda = \mu = 1$, $Pf(x) = f(0) + f'(0)x$.

In all the above eight forms of the projection P, one can verify that P is Hermitian.

Case II. Suppose T is form $(\star\star)$, then

$$\begin{aligned} &\lambda\mu f(0) + \lambda\mu f'(0)x + \int_0^x \int_0^t \omega(s)\omega(\phi(s))f''(\phi^2(s))dsdt \\ &- (1+\alpha)\left(\lambda f'(0) + \mu f(0)x + \int_0^x \int_0^t \omega(s)f''(\phi(s))\ ds\ dt\right) + \alpha f(x) = 0. \end{aligned} \tag{4.2.3}$$

For $f = 1$ Equation (4.2.3) reduces to $\lambda\mu - (1+\alpha)\mu x + \alpha = 0$ or $\lambda\mu = -\alpha$. This implies that $\alpha = -1$.

If we differentiate Equation (4.2.3) twice we get $\omega(x)\omega(\phi(x))f''(\phi^2(x)) = f''(x)$. Thus, $x = \phi^2(x)$, and hence $\omega(x)\omega(\phi(x)) = 1$. $\square$

Corollary 4.2.4. *A projection P on $C^2[0,1]$ is Hermitian if and only if there exist real constants a, b and c such that $Pf(x) = af(x) + bf(0) + cf'(0)x$ for all $f \in C^2[0,1]$ and $x \in [0,1]$, where*

$$(a,b,c) \in \{(1,-1,-1), (1,-1,0), (1,0,-1), (1,0,0), (0,0,0), (0,0,1), (0,1,0), (0,1,1)\}.$$

4.3 Relationship of generalized bi-circular and Hermitian projections with average of isometries on $C^2[0,1]$

The following theorem describes the relationship of GBPs and Hermitian projections with average of surjective isometries on $C^2[0,1]$.

Theorem 4.3.1. *The average of two surjective linear isometries T_1 and T_2 on $C^2[0,1]$ is a projection P if and only if either P is Hermitian or $P = \frac{I+T}{2}$, for some isometric reflection T.*

Proof. The *'if'* part is obvious. For the *'Only if'* part, let $P = \frac{1}{2}(T_1 + T_2)$. Consider the following partition of $[0,1]$:

$X_1 = \{x \in [0,1] : x = \phi_1(x) = \phi_2(x),\ \omega_1(x) = \omega_2(x) = 1\}$,

$X_2 = \{x \in [0,1] : \phi_1(x) = \phi_2(x),\ \omega_1(x) = -\omega_2(x)\}$,

$X_3 = \{x \notin X_1 \cup X_2 : \phi_1(x) = x,\ \phi_2^2(x) = x,\ \phi_1 \circ \phi_2(x) = \phi_2(x), \omega_1(x) = \omega_1(\phi_2(x)) = 1, \omega_2(x)\omega_2(\phi_2(x)) = 1\}$,

$X_4 = \{x \notin X_1 \cup X_2 : \phi_2(x) = x,\ \phi_1^2(x) = x,\ \phi_2 \circ \phi_1(x) = \phi_1(x), \omega_2(x) = \omega_2(\phi_1(x)) = 1, \omega_1(x)\omega_1(\phi_1(x)) = 1\}$.

We consider the following cases.

Case I. Let $X_1 \cup X_2 = [0,1]$.

If $P = \frac{1}{2}(F_1 + F_2)$. From Lemma 4.1.4 we have the following four cases:

1. $\lambda_1 + \lambda_2 = 0$, $\mu_1 + \mu_2 = 0$. In this case, if $x \in X_1$, then we have

$$\begin{aligned} Pf(x) = \frac{1}{2}(F_1 + F_2)f(x) &= \frac{1}{2}\big[\lambda_1 f(0) + \mu_1 f'(0)x + (\zeta^2(\omega_1(f'' \circ \phi_1)))(x) \\ &\qquad + \lambda_2 f(0) + \mu_2 f'(0)x + (\zeta^2(\omega_2(f'' \circ \phi_2)))(x)\big] \\ &= \frac{1}{2}\int_0^x \int_0^t f''(s)dsdt + \frac{1}{2}\int_0^x \int_0^t f''(s)dsdt \\ &= f(x) - f(0) - f'(0)x. \end{aligned}$$

If $x \in X_2$, then

$$\begin{aligned} Pf(x) = \frac{1}{2}(F_1 + F_2)f(x) &= \frac{1}{2}\big[\lambda_1 f(0) + \mu_1 f'(0)x + (\zeta^2(\omega_1(f'' \circ \phi_1)))(x) \\ &\qquad + \lambda_2 f(0) + \mu_2 f'(0)x + (\zeta^2(\omega_2(f'' \circ \phi_2)))(x)\big] \\ &= 0. \end{aligned}$$

2. $\lambda_1 + \lambda_2 = 0$, $\mu_1 = \mu_2 = 1$. Here, we have

$$Pf(x) = \begin{cases} f(x) - f(0), & x \in X_1 \\ f'(0)x, & x \in X_2. \end{cases}$$

3. $\lambda_1 = \lambda_2 = 1$, $\mu_1 + \mu_2 = 0$. Thus,

$$Pf(x) = \begin{cases} f(x) - f'(0)x, & x \in X_1 \\ f(0), & x \in X_2. \end{cases}$$

4. $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \mu_2 = 1$. This implies that

$$Pf(x) = \begin{cases} f(x), & x \in X_1 \\ f(0) + f'(0)x, & x \in X_2. \end{cases}$$

Corollary 4.2.4 implies that P in all the above cases are Hermitian.

If $P = \frac{1}{2}(S_1 + S_2)$, then $\lambda_1 + \lambda_2 = 0$ and $\mu_1 + \mu_2 = 0$, see Lemma 4.1.6. The projection in this case will be Hermitian. To be precise,

$$Pf(x) = \begin{cases} f(x) - f(0) - f'(0)x, & x \in X_1 \\ 0, & x \in X_2. \end{cases}$$

Case II. Let $X_1 \cup X_2 \neq [0,1]$. Define $\phi : [0,1] \to [0,1]$ as

$$\phi(x) = \begin{cases} x, & x \in X_1 \cup X_2 \\ \phi_2(x), & x \in X_3 \\ \phi_1(x), & x \in X_4, \end{cases}$$

and $\omega : [0,1] \to \mathbb{T}$ as $\omega(x) = \omega_1(x) + \omega_2(x) - 1$.

We observe that ϕ is a homeomorphism. Indeed, the continuity of ϕ_1 and ϕ_2 imply that ϕ is continuous. Moreover, using conditions of ϕ_1 and ϕ_2 in the sets X_i, $i = 1,2,3,4$, we can prove that ϕ is a bijection. Since $[0,1]$ is compact, ϕ is a homeomorphism. Furthermore, it is also clear that ω is a continuous modulus 1 function.

We will show that there exists an isometry T on $C^2[0,1]$ such that $P = \frac{T_1+T_2}{2} = \frac{I+T}{2}$. In particular, P is a GBP.

Let $P = \frac{1}{2}(F_1 + F_2)$. The following cases may occur (see Lemma 4.1.4).

1. $\lambda_1 + \lambda_2 = 0$, $\mu_1 + \mu_2 = 0$,
2. $\lambda_1 + \lambda_2 = 0$, $\mu_1 = \mu_2 = 1$,
3. $\lambda_1 = \lambda_2 = 1$, $\mu_1 + \mu_2 = 0$,
4. $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \mu_2 = 1$.

Let

$$(\lambda, \mu) = (-1,-1),\ (-1,1),\ (1,-1) \text{ and } (1,1) \tag{4.3.1}$$

in (1), (2), (3) and (4) above, respectively. For $f \in C^2[0,1]$ and $x \in [0,1]$, define

$$Ff(x) = \lambda f(0) + \mu f'(0)x + (\zeta^2(\omega(f'' \circ \phi)))(x).$$

Suppose Case (1) holds, that is, $\lambda_1 + \lambda_2 = 0$, $\mu_1 + \mu_2 = 0$ and $(\lambda, \mu) = (-1,-1)$. We will show that $F_1 + F_2 = I + F$.

Let $x \in X_1$. Then $\phi(x) = x$ and $\omega(x) = 1$. Now,

$$\begin{aligned}\frac{(I+F)f(x)}{2} &= \frac{1}{2}\big[f(x) - f(0) - f'(0)x + (\zeta^2(\omega(f'' \circ \phi)))(x)\big]\\ &= f(x) - f(0) - f'(0)x.\end{aligned}$$

If $x \in X_2$, then $\phi(x) = x$ and $\omega(x) = -1$. Thus,

$$\frac{(I+F)f(x)}{2} = f(x) - f(0) - f'(0)x + (\zeta^2(\omega(f'' \circ \phi)))(x) = 0.$$

If $x \in X_3$, then $\phi(x) = \phi_2(x)$ and $\omega(x) = \omega_2(x)$. Hence,

$$\begin{aligned}(F_1+F_2)f(x) &= \lambda_1 f(0) + \mu_1 f'(0)x + (\zeta^2(\omega_1(f''\circ\phi_1)))(x)\\ &\qquad + \lambda_2 f(0) + \mu_2 f'(0)x + (\zeta^2(\omega_2(f''\circ\phi_2)))(x)\\ &= \int_0^x\int_0^t f''(s)dsdt + \int_0^x\int_0^t \omega_2(s)f''(\phi_2(s))dsdt\\ &= f(x) - f(0) - f'(0)x + \int_0^x\int_0^t \omega_2(s)f''(\phi_2(s))dsdt.\end{aligned}$$

Now,

$$\begin{aligned}(I+F)f(x) &= f(x) - f(0) - f'(0)x + (\zeta^2(\omega(f''\circ\phi)))(x)\\ &= f(x) - f(0) - f'(0)x + \int_0^x\int_0^t \omega_2(s)f''(\phi_2(s))dsdt.\end{aligned}$$

This implies that $Pf(x) = \frac{F_1f(x)+F_2f(x)}{2} = \frac{f(x)+Ff(x)}{2}$ for all $f \in C^2[0,1]$ and $x \in [0,1]$.

The case when $x \in X_4$ is similar.

Proceeding exactly as above, we can show that in Cases (2), (3) and (4), the chosen values of λ and μ in (4.3.1) will imply that $\frac{F_1+F_2}{2} = \frac{I+F}{2}$.

Let $P = \frac{1}{2}(F+S)$. Lemma 4.1.5 implies that $\lambda_1 = \mu_1 = 1$ and $\lambda_2\mu_2 = 1$. For $f \in C^2[0,1]$ and $x \in [0,1]$, define

$$S'f(x) = \lambda_2 f'(0) + \overline{\lambda_2} f(0)x + (\zeta^2(\omega(f''\circ\phi)))(x).$$

Let $x \in X_1$. We have

$$\begin{aligned}(F+S)f(x) &= f(0) + f'(0)x + (\zeta^2(\omega_1(f''\circ\phi_1)))(x)\\ &\qquad + \lambda_2 f'(0) + \overline{\lambda_2} f(0)x + (\zeta^2(\omega_2(f''\circ\phi_2)))(x)\\ &= 2f(x) + \lambda_2 f'(0) - f(0) + (\overline{\lambda_2} f(0) - f'(0))x.\end{aligned}$$

Now,

$$\begin{aligned}(I+S')f(x) &= f(x) + \lambda_2 f'(0) + \overline{\lambda_2} f(0)x + (\zeta^2(\omega(f''\circ\phi)))(x)\\ &= 2f(x) + \lambda_2 f'(0) - f(0) + (\overline{\lambda_2} f(0) - f'(0))x.\end{aligned}$$

If $x \in X_2$, then

$$(F+S)f(x) = f(0) + f'(0)x + \lambda_2 f'(0) + \overline{\lambda_2} f(0)x.$$

Further,

$$\begin{aligned}(I+S')f(x) &= f(x) + \lambda_2 f'(0) + \overline{\lambda_2} f(0)x - f(x) + f(0) + f'(0)x \\ &= f(0) + f'(0)x + \lambda_2 f'(0) + \overline{\lambda_2} f(0)x.\end{aligned}$$

For $x \in X_3$,

$$\begin{aligned}(F+S)f(x) = f(0) + f'(0)x &+ \int_0^x \int_0^t f''(s)dsdt \\ &+ \lambda_2 f'(0) + \overline{\lambda_2} f(0)x + \int_0^x \int_0^t \omega_2(s) f''(\phi_2(s))dsdt \\ = f(x) &+ \lambda_2 f'(0) + \overline{\lambda_2} f(0)x + \int_0^x \int_0^t \omega_2(s) f''(\phi_2(s))dsdt.\end{aligned}$$

Also,

$$(I+S')f(x) = f(x) + \lambda_2 f'(0) + \overline{\lambda_2} f(0)x + \int_0^x \int_0^t \omega_2(s) f''(\phi_2(s))dsdt.$$

The case of $x \in X_4$ is exactly similar.

All these considerations imply that $\frac{F+S}{2} = \frac{I+S'}{2}$.

Let $P = \frac{1}{2}(S_1 + S_2)$. Then $\lambda_1 + \lambda_2 = 0$ and $\mu_1 + \mu_2 = 0$, see Lemma 4.1.6. By repeating the above process, it can be easily shown that $\frac{S_1+S_2}{2} = \frac{I+F}{2}$, where

$$Ff(x) = -f(0) - f'(0)x + (\zeta^2(\omega(f'' \circ \phi)))(x).$$

This completes the proof. $\square$

CHAPTER 5

Conclusion, challenges and some future plans

5.1 Conclusion and challenges

In this thesis, we have worked mainly on two problems. One is the problem of algebraic reflexivity of set of isometries on some Banach spaces. We have shown that in many important cases the local maps (isometries in our case) in consideration are all global, i.e., they belong to the given class of operators. We observe that we have defined the above problem for linear algebraic structures only. Moreover, the local maps are also linear. It is natural to think about the above problems for more general structures.

The second is the problem of characterizing some special classes of norm-one projections on the space $C^2[0,1]$. We have also studied the relation between isometries and projections in this space. Although our proofs in chapter 4 suggest that similar results should be true for the space $C^r[0,1]$, the number of cases occurring becomes difficult to handle, especially, for $r \geq 4$. This is also the case if we take projections as a convex combination of 4 isometries or more. So, a different approach is needed to handle the general case.

5.2 Future plans

In this section, we mention some research problems for future work.

5.2.1 Local isometries on subspaces of vector-valued function spaces

The algebraic reflexivity problems discussed for subspaces of $C_0(X)$ could asked for certain subspaces of vector-valued function spaces.

Let $C_0(X, E)$ be the Banach space of E-valued continuous functions on X vanishing at infinity and endowed with the supremum norm $\|.\|_\infty$. We denote by $S_E = \{e \in E : \|e\| = 1\}$, the unit sphere of E. For $f \in C_0(X)$ and $e \in E$, we define the map $f \otimes e : X \to E$ by $(f \otimes e)(x) = f(x)e$. We can easily verify that $f \otimes e \in C_0(X, E)$.

Definition 5.2.1. *Let A be a subspace of $C_0(X)$. We denote by $\mathcal{A}[A]$ any subspace of $C_0(X, E)$ which contains the set $\{f \otimes e : f \in A, e \in S_E\}$.*

Font [29] characterized the structure of linear isometries of $\mathcal{A}[A]$ onto such a subspace $\mathcal{B}[B]$ of $C_0(Y, F)$, where A and B are regular closed subalgebras of $C_0(X)$ and $C_0(Y)$ respectively, and E and F are strictly convex Banach spaces. We shall investigate the algebraic reflexivity problem for the set of all surjective linear isometries between the subspaces $\mathcal{A}[A]$ and $\mathcal{B}[B]$ of $C_0(X, E)$ and $C_0(Y, F)$ respectively.

5.2.2 Algebraic reflexivity in the non-linear case

One of the simple ideas to generalize the reflexivity problems to the non-linear case is due to Šemrl [52]. Let $\mathcal{A}$ be any mathematical structure, and let $\mathcal{E}$ be a given class of transformations on $\mathcal{A}$. We say that a map $\phi : \mathcal{A} \to \mathcal{A}$ belongs 2-locally to $\mathcal{E}$ if for any pair $x, y \in \mathcal{A}$, there is an element $\phi_{(x,y)}$ of $\mathcal{E}$ for which $\phi(x) = \phi_{(x,y)}(x)$ and $\phi(y) = \phi_{(x,y)}(y)$. Adopting the definition of algebraic reflexivity for linear (1-) local maps, we call the class $\mathcal{E}$ algebraically reflexive, if for every map ϕ that belongs 2-locally to $\mathcal{E}$, we necessarily have $\phi \in \mathcal{E}$.

We are interested to study the following problem proposed by Prof. Lajos Molnár.

Let $\mathcal{H}$ be a Hilbert space, and let $B_s(\mathcal{H})$ be the set of all self-adjoint operators on $\mathcal{H}$. We equip the set $B_s(\mathcal{H})$ with the usual order, i.e., for any $A, B \in B_s(\mathcal{H})$, we write $A \leq B$ if $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ holds for every $x \in \mathcal{H}$.

A bijective map $\Phi : B_s(\mathcal{H}) \to B_s(\mathcal{H})$ is called an order-automorphism if it preserves the order, i.e., $A \leq B \iff \Phi(A) \leq \Phi(B)$. The structure of the group of all order-automorphisms of $B_s(\mathcal{H})$ is characterized in [46]. The problem is to investigate whether the group of all order-automorphisms of $B_s(\mathcal{H})$ is algebraically reflexive or not.

5.2.3 Generalized n-circular projections on Banach spaces

The notion of generalized bi-circular projection was generalized in [1, 3] as follows.

Definition 5.2.2. *A projection P_0 on a Banach space E is said to be a generalized n-circular projection, $n \geq 2$, if there exist $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{T} \setminus \{1\}$, λ_i, $i = 1, 2, \ldots, n-1$ of finite order and non-trivial projections $P_1, P_2, \ldots, P_{n-1}$ on E such that*

1. *$\lambda_i \neq \lambda_j$ for $i \neq j$,*

2. *$P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$,*

3. *$P_0 + \lambda_1 P_1 \cdots + \lambda_{n-1} P_{n-1}$ is a surjective isometry.*

Many authors studied generalized 3-circular projections (also called generalized tri-circular projections) on several Banach spaces, see for example [1, 3, 4, 24] and [35].

A complete characterization of the structure of generalized n-circular projections on classical Banach spaces seems to be complicated with available techniques. We want to study this problem for $n = 3$ or more for some specific spaces, as for example $L^p(\Omega, E)$, $1 \leq p < \infty$, $p \neq 2$, where (Ω, μ) is a σ-finite measure space, and E is a separable Banach space with trivial L^p-structure.

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