# TAYLOR'S THEOREM

(1) Let $f : [a, b] \longrightarrow \mathbb{R}$ and $n$ be a non-negative integer. Suppose that $f^{(n+1)}$ exists on $[a, b]$. Show that $f$ is a polynomial of degree $\leq n$ if $f^{(n+1)}(x) = 0$ for all $x \in [a, b]$. Observe that the statement for $n = 0$ can be proved by the mean value theorem.

(2) Show that $1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1 + x} \leq 1 + \frac{x}{2}$ for $x > 0$.

(3) Show that for $x \in \mathbb{R}$ with $|x|^5 < \frac{5!}{10^4}$, we can replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitude less than or equal to $10^{-4}$.

(4) Prove the binomial expansion: $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + x^n, \ x \in \mathbb{R}$.

(5) Using Taylor's theorem compute: $\lim_{x \to 0} \frac{1 - \sqrt{1 - x^2} \cos x}{x^4}$.

(6) If $x \in [0, 1]$ and $n \in \mathbb{N}$, show that

$$\left| \ln(1 + x) - \left( x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^n}{n} \right) \right| < \frac{x^{n+1}}{n + 1}.$$

(7) (a) Let $f : [a, b] \longrightarrow \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in [a, b]$. Suppose $x_0 \in [a, b]$. Show that for any $x \in [a, b]$

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

i.e., the graph of $f$ lies above the tangent line to the graph at $(x_0, f(x_0))$.

(b) Show that $\cos y - \cos x \geq (x - y) \sin x$ for all $x, y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.

(8) (a) Let $f : [a, b] \longrightarrow \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in [a, b]$. Suppose $x, y \in (a, b), x < y$ and $0 < \lambda < 1$. Show that

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

i.e., the chord joining the two points $(x, f(x))$ and $(y, f(y))$ lies above the portion of the graph $\{(z, f(z)) : z \in (x, y)\}$.

(b) Show that $\lambda \sin x \leq \sin \lambda x$ for all $x \in [0; \pi]$ and $0 < \lambda < 1$.

(9) Let $f$ be a twice differentiable function on $\mathbb{R}$ such that $f''(x) \geq 0$ for all $x \in \mathbb{R}$. Show that if $f$ is bounded then it is a constant function.

(10) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that $f'''(x) > 0$ for all $x \in \mathbb{R}$. Suppose that $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$. Show that $f(x_2) - f(x_1) > f'\left(\frac{x_1 + x_2}{2}\right)(x_2 - x_1)$.

(11) Suppose $f$ is a three times differentiable function on $[-1, 1]$ such that $f(-1) = 0$, $f(1) = 1$ and $f'(0) = 0$. Using Taylor's theorem show that $f'''(c) \geq 3$ for some $c \in (-1, 1)$.