# DIFFERENTIABILITY

(1) Discuss the differentiability of the following functions at $x = 0$.

(a) $f(x) = x^{\frac{1}{3}}$

(b) $f(x) = x^2$ for rational $x$ and $f(x) = 0$ for irrational $x$.

(c) $f(x) = x \sin x \cos \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

(d) Let $m, n$ be positive integers. Define

$$f(x) = \begin{cases} x^n, & x \geq 0 \\ x^m, & x < 0. \end{cases}$$

(2) Give an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is differentiable only at $x = 1$.

(3) Let $n \in \mathbb{N}$. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = x^n$ for $x \geq 0$ and $f(x) = 0$ if $x < 0$. For which values of $n$,

(a) is $f$ continuous at 0?

(b) is $f$ differentiable at 0?

(c) is $f'$ continuous at 0?

(d) is $f'$ differentiable at 0?

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x)| \leq x^2$ for all $x \in \mathbb{R}$. Discuss the differentiability of $f$ at 0.

(5) Find the number of real solutions of the following equations.

(a) $2x - \cos^2 x + \sqrt{7} = 0$.

(b) $x^{17} - e^{-x} + 5x + \cos x = 0$.

(6) Let $P(X) := \sum_{k=0}^{n} a_k X^k$, $n \geq 2$ be a real polynomial. Assume that all roots of $P$ lie in $\mathbb{R}$. Show that all roots its derivative $P'(X)$ are also real.

(7) Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = 0$. Show that the polynomial $q(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}$ has at least one real root.

(8) Let $f : [a, b] \longrightarrow \mathbb{R}$ be such that $f'''(x)$ exists for all $x \in [a, b]$. Suppose $f(a) = f(b) = f'(a) = f'(b) = 0$ Show that the equation $f'''(x) = 0$ has a solution.

(9) Let $f : (a, b) \to \mathbb{R}$ be a function. Prove that $f$ is differentiable at $x$ if and only if there exists a (unique) linear map $A : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{f(x + h) - f(x) - Ah}{|h|} = 0.$$

(10) (**Differentiable Inverse Theorem**). Suppose $f : J \to \mathbb{R}$ is a one-one and continuous function. If $f$ is differentiable at $c$ and $f'(c) \neq 0$, then $f^{-1} : f(J) \to J$ is differentiable at $f(c)$ and $(f^{-1})'(f(c)) = \frac{1}{f'(c)}$.