PROPERTIES OF CONTINUOUS FUNCTIONS

(1) Let $f : J \to \mathbb{R}$ be a strictly monotonic function such that $f(J)$ is an interval. Show that f is continuous.

(2) Let $f : [a, b] \to [a, b]$ be continuous. Show that f has a fixed point in $[a, b]$, i.e., $\exists\, c \in [a, b]$ such that $f(c) = c$.

(3) Prove that $x = \cos x$ for some $x \in (0, \pi/2)$.

(4) Prove that $xe^x = 1$ for some $x \in (0, 1)$.

(5) Is there a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) \notin \mathbb{Q}$ for $x \in \mathbb{Q}$ and $f(x) \in \mathbb{Q}$ for $x \notin \mathbb{Q}$.

(6) Show that a polynomial of odd degree has at least one real root.

(7) Show that $x^4 + 5x^3 - 7$ has at least two real roots.

(8) Let $p(X) := a_0 + a_1X + \cdots + a_nX^n$, n is even. If $a_0a_n < 0$, show that p has at least two real roots.

(9) Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Show that $f([a, b]) = [c, d]$ for some $c, d \in \mathbb{R}$ with $c \leq d$. Can you identify c, d.

(10) Construct a continuous bijection $f : [a, b] \longrightarrow [c, d]$ such that f^{-1} is continuous.

(11) Construct a continuous function from $(0, 1)$ onto $[0, 1]$. Can such a function be one-one.

(12) Construct a continuous one-one function from $(0, 1)$ to $[0, 1]$. Can such a function be onto.