CONTINUITY AND LIMITS

(1) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $f(c) > 0$ for some $c \in \mathbb{R}$. Show that there exists an $\epsilon > 0$ such that $f(x) > 0$ for all $x \in (c - \epsilon, c + \epsilon)$.

(2) Let $f : (-1, 1) \to \mathbb{R}$ be a continuous function such that in every neighborhood of 0, there exists a point where f takes the value 0. Show that $f(0) = 0$.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous everywhere.

(4) Discuss the continuity/discontinuity for the following functions

(a) $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 3x + 1, & \text{if } x \in \mathbb{Q} \\ x, & \text{otherwise.} \end{cases}$$

(b) $f : [0, \pi] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}, & \text{otherwise.} \end{cases}$$

(5) Let A be a nonempty subset of $\mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \inf\{|x - a| : a \in A\}$. Show that f is continuous on $\mathbb{R}$.

(6) Let $J = \{\frac{1}{n} : n \in \mathbb{N}\}$. Show that any function $f : J \longrightarrow \mathbb{R}$ is continuous on J.

(7) A function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be Lipschitz on $[a, b]$ if there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in [a, b]$. Show that any Lipschitz function is continuous.

(8) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous such that given any two points $x < y$, there exists a point z such that $x < z < y$ and $f(z) = g(z)$. Show that $f(x) = g(x)$ for all x.

(9) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. If $f(x) = g(x)$ for $x \in \mathbb{Q}$, then show that $f = g$.

(10) Let $f : [0, \infty) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1, & \text{if } x = 0, \\ 1/q, & \text{if } x = p/q, \text{ where } p, q \in \mathbb{N} \text{ and } p, q \text{ have no common factors }, \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is continuous at every irrational in $[0, \infty)$.