PROBLEM SET 03: CAUCHY SEQUENCES AND SUBSEQUENCE

(1) Prove that the sum of two Cauchy sequences is Cauchy.

(2) Prove that the product of two Cauchy sequences is Cauchy.

(3) Let (x_n) be a sequence and let $a > 1$. Assume that $|x_{k+1} - x_k| < a^{-k}$ for all $k \in \mathbb{N}$. Show that (x_n) is Cauchy.

(4) Let (x_n) be defined by

$$x_n = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{(-1)^{n+1}}{n!}.$$

Show that (x_n) converges.

(5) Show that the sequences (x_n) defined below are Cauchy.

(a) $x_1 = 1$, $x_{n+1} = \frac{1}{2+x_n^2}$.

(b) $x_1 = 1$, $x_{n+1} = \frac{1}{6}(x_n^2 + 8)$.

(c) $x_1 = \frac{1}{2}$, $x_{n+1} = \frac{1}{7}(x_n^3 + 2)$.

(d) $x_1 = a$, $x_2 = b$, $x_{n+2} = \frac{x_n + x_{n+1}}{2}$, where a and b are two distinct real numbers.

(e) Let $0 < a \le x_1 \le x_2 \le b$, $x_{n+2} = \sqrt{x_{n+1}x_n}$.

(6) Show that the following sequences cannot converge.

(a) $x_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$.

(b) $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

(7) Let (x_n) be a sequence such that $|x_n| \le \frac{1+n}{1+n+2n^2}$ for all n. Prove that (x_n) is Cauchy.

(8) Show that if a monotone sequence has a convergent subsequence, then it is convergent.

(9) Let (r_n) be an enumeration of all rational numbers in $[0, 1]$. Show that (r_n) is not convergent.

(10) **(Elaborate version of Bolzano-Weierstrass Theorem)** Let (x_n) be a sequence. If (x_n) is bounded above and does not diverge to $-\infty$, then prove that (x_n) has a convergent subsequence. Likewise, if (x_n) is bounded below and does not diverge to ∞, then prove that (x_n) has a convergent subsequence.