PROBLEM SET 02: SEQUENCES AND THEIR CONVERGENCE

(1) Let $x_n \to \ell$. Show that if we alter a finite number of terms of (x_n), the new sequence still converges to ℓ.

(2) Let $x_n \to \ell$. If $\ell > 0$, then show that except for a finite number of terms, all $x_n > 0$.

(3) Let $x_n \leq y_n$ for all n such that $x_n \to x$ and $y_n \to y$. Show that $x \leq y$.

(4) True or False:

(a) If (x_n) and $(x_n y_n)$ are bounded, then (y_n) is bounded.

(b) If (x_n) and (y_n) are such that $x_n y_n \to 0$, then one of the sequences converges to 0.

(5) In each of the following sequences, write the first following terms of (x_n), and then investigate its convergence.

(a) $x_n = \frac{n^r}{(1+s)^n}$, where $r, s > 0$.

(b) $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} \cdots + \frac{1}{\sqrt{n^2+n}}$.

(c) $x_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+n+n}$.

(d) $x_n = a^n(2n)^b$ where $0 < a < 1$ and $b > 1$.

(e) $x_n = (a^n + b^n)^{1/n}$ where $0 < a < b$.

(f) $x_n = n^\alpha - (n+1)^\alpha$ for some $\alpha \in (0, 1)$.

(6) Show that the sequence (x_n) is bounded and monotone, and find its limit where

(a) $x_1 = 1$ and $x_{n+1} = \sqrt{3x_n}$.

(b) $x_1 = 1$ and $x_{n+1} = \frac{4+3x_n}{3+2x_n}$.

(7) Let $M = \sup A \subset \mathbb{R}$. Show that there exists a sequence (x_n) in A such that $x_n \to M$. Prove the analogous result for infimum as well.

(8) Show that the sequence $x_n = \sum_{k=1}^{n} \frac{1}{k}$ diverges to ∞.

(9) Let $0 < y_1 < x_1$. For $n \geq 2$, define

$$x_{n+1} = \frac{x_n + y_n}{2} \text{ and } y_{n+1} = \sqrt{x_n y_n}.$$

(a) Prove that (y_n) is increasing and bounded above while (x_n) is decreasing and bounded below.

(b) For $n \in \mathbb{N}$, prove that

$$0 < x_{n+1} - y_{n+1} < \frac{1}{2^n}(x_1 - y_1).$$

(c) Prove (x_n) and (y_n) converges to the same limit.

(10) (**Euler's number** e) In general, it may not be possible to explicitly find the limit of a convergent sequence. So, some real numbers are defined as the limit of such sequences.

Let $x_n = (1 + \frac{1}{n})^n$. Prove that (x_n) is increasing and bounded above. Define $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$. Show that e is irrational.