FUNCTION OF SEVERAL VARIABLES

(1) Show that the limit exists for the following

(a) $\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^2+y^2}$

(b) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$

(c) $\lim_{(x,y)\to(0,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2}$

(2) Show that the limit does not exist for the following

(a) $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$

(c) $\lim_{(x,y)\to(0,0)} \frac{xy\cos(y)}{4x^2+y^2}$

(3) For

$$f(x,y) = \begin{cases} xy\tan(y/x), \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0), \end{cases}$$

show that $xf_x + yf_y = 2f$.

(4) Discuss the continuity, partial derivatives, directional derivative in any direction and differentiability of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ at origin:

(a) $f(x,y) = |x| + |y|$;

(b) $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2}, \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0) \end{cases}$;

(c) $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0) \end{cases}$;

(d) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0) \end{cases}$;

(e) $f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y}, \text{ if } x \neq y \\ 0, \text{ if } x = y \end{cases}$;

(f) $f(x,y) = \begin{cases} y\sin(1/x), \text{ if } x \neq 0 \\ y, \text{ if } x = 0 \end{cases}$;

(g) $f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y), \text{ if } xy \neq 0 \\ x\sin(1/x), \text{ if } x \neq 0, y = 0 \\ y\sin(1/y), \text{ if } y \neq 0, x = 0 \\ 0, \text{ if } x = 0, y = 0 \end{cases}$.

(5) Examine the equality of $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the following functions:

(a) $f(x,y) = |x^2 - y^2|$;

(b) $f(x,y) = \begin{cases} (x^2+y^2)\tan^{-1}(y/x), & \text{if } x \neq 0 \\ \pi y^2/2, & \text{if } x = 0 \end{cases}$;