FTC, RIEMANN SUM AND IMPROPER INTEGRALS

(1) (a) Show that every continuous function on a closed bounded interval is a derivative.
(b) Show that and integrable function on a closed bounded interval need not be a derivative.

(2) (a) Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = 0$ for $-1 \le x < 0$ and $f(x) = 1$ for $0 \le x \le 1$/ Define $F(x) = \int_{-1}^{x} f(t)dt$
(i) Sketch the graphs of f and F and observe that f is not continuous; however, F is continuous.
(ii) Observe that F is not differentiable at 0.
(b) Give an example of a function f on $[-1, 1]$ such that f is not continuous at 0 but $F(x)$ defined by $F(x) = \int_{-1}^{x} f(t)dt$ is differentiable at 0.

(3) Prove that second FTC by assuming the integrand to be continuous.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Define $g(x) = \int_0^x (x - t)f(t)dt$ for all $x \in \mathbb{R}$. Show that $g'' = f$.

(5) Let f be a differentiable function on $[0, 1]$. Show that there exists $c \in (0, 1)$ such that $\int_0^1 f(x)dx = f(0) + \frac{1}{2}f'(c)$.

(6) Let $f : [0, a] \to \mathbb{R}$ be such that $f''(x) > 0$ for every $x \in [0, a]$. Show that $\int_0^a f(x)dx \ge af(\frac{a}{2})$.

(7) Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions. Suppose that f is increasing and g is non-negative on $[a, b]$. Show that there exists $c \in [a, b]$ such that $\int_a^b f(x)g(x)dx = f(b)\int_a^c g(x)dx + f(a)\int_c^b g(x)dx$.

(8) Show that $\frac{\pi^2}{9} \le \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} \le \frac{2\pi^2}{9}$.

(9) Let $f : [0, 1] \to \mathbb{R}$ be continous. Show that $\lim_{n\to\infty} \int_0^1 f(x^n)dx = f(0)$.

(10) Show that $\lim_{n\to\infty} \frac{1}{n^3}\left[\sin\frac{\pi}{n} + 2^2\sin\frac{2\pi}{n} + \cdots + n^2\sin\frac{n\pi}{n}\right] = \int_0^1 x^2\sin(\pi x)dx$.

(11) Let $a_n = \ln\left(\frac{(n!)^{\frac{1}{n}}}{n}\right)$ for all $n \in \mathbb{N}$. Convert a_n into a Riemann sum and find $\lim_{n\to\infty} a_n$.

(12) Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = \int_1^x \frac{\ln t}{1+t}dt$. Solve the equation $f(x) + f(\frac{1}{x}) = 2$.

(13) Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(t) = \frac{(-1)^{n+1}}{n}$ when $t \in [n - 1, n)$, $n \in \mathbb{N}$. Show that $\int_0^\infty f(t)dt$ converges but not absolutely.

(14) Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(n) = 1$ for all $n \in N$ and $f(x) = 0$ if $x \in [1, \infty) \setminus \mathbb{N}$. Then show that
(a) $\int_1^\infty f(t)dt$ converges but $\sum_{n=1}^\infty f(n)$ diverges.
(b) $\int_1^\infty (f(t) - 1)dt$ diverges but $\sum_{n=1}^\infty (f(n) - 1)$ converges.

(15) (a) Let $f : [1, \infty) \to \mathbb{R}$ be such that $f(n) = 1$ for all $n\mathbb{N}$ and $f(t) = 0$ otherwise. Show that $\int_1^\infty f(t)dt$ converges but $f(t) \nrightarrow 0$ as $t \to \infty$.

(b) Does there exist a continuous function $f : [1, \infty) \to \mathbb{R}$ such that $\int_1^\infty f(t)dt$ converges but $f(t) \nrightarrow 0$ as $t \to \infty$?

(16) Determine the convergence/divergence of the following integrals.

(a) $\int_1^\infty \frac{e^t}{4^t} dt$ (b) $\int_1^\infty t \sin t^4 dt$ (c) $\int_1^\infty \frac{\sqrt{t}}{e^{\sin t - 1}} dt$ (d) $\int_1^\infty \frac{1 - 5\sin 2t}{t^2 + \sqrt{t}} dt$