RIEMANN INTEGRATION

(1) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose that there is a partition P of $[a, b]$ such that $L(P, f) = U(P, f)$. Show that f is a constant function.

(2) Let $f : [a, b] \to \mathbb{R}$ be a integrable function and P_n be a partition such that $U(P_n, f) - L(P_n, f) \to 0$. Show that $\lim\limits_{n\to\infty} U(P_n, f) = \lim\limits_{n\to\infty} L(P_n, f) = \int\limits_a^b f(x)\ dx$.

(3) Let $f : [0, 1] \to \mathbb{R}$ be a function defined as $f(x) = \frac{1}{2^n}$, when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$ $(n = 0, 1, 2, \ldots)$ and $f(0) = 0$. Show that f is integrable and find its integral value.

(4) Consider $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = a$, $-1 \le x < 0$, $f(0) = 0$, and $f(x) = b$, $0 < x \le 1$. Show that f is integrable.

(5) Determine whether the function $f : [0, 1] \to \mathbb{R}$ defined as $f(x) = \sin \frac{1}{x}$, if x is irrational and $f(x) = 0$, if x is rational, is Riemann integrable.

(6) Let $f : [a, b] \to \mathbb{R}$ be a non-negative continuous function such that $\int\limits_a^b f(x)\ dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

(7) Give an example of an integrable non-negative function f on $[a, b]$ such that $\int\limits_a^b f(x)\ dx = 0$ but $f(c) \neq 0$ for some $c \in [a, b]$.

(8) Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions.
   (a) Show that $|f|, f^2, fg, \max\{f, g\}$ and $\min\{f, g\}$ are integrable.
   (b) Show that $|\int\limits_a^b f(x)\ dx| \le \int\limits_a^b |f(x)|\ dx$.
   (c) If $f(x) \le g(x)$ for all $x \in [a, b]$, show that $\int\limits_a^b f(x)\ dx \le \int\limits_a^b g(x)\ dx$.

(9) Give an example of a function $f : [a, b] \to \mathbb{R}$ such that $|f|$ is integrable but f is not integrable.

(10) Give an example of a function $f : [a, b] \to \mathbb{R}$ such that f^2 is integrable but f is not integrable.