POWER SERIES

(1) For a given $\sum_{n=0}^{\infty} a_n x^n$, let

$$K = \{|x| : x \in \mathbb{R} \text{ and } \sum_{n=0}^{\infty} a_n x^n \text{ is convergent } \}$$

be bounded. If $r = \sup K$, then $\sum_{n=0}^{\infty} a_n x^n$

(a) converges absolutely for all $x \in \mathbb{R}$ with $|x| < r$,

(b) diverges for all $x \in \mathbb{R}$ with $|x| > r$.

(2) In each of the following cases, determine the radius of convergence and the values of x for which the power series converges:

(a) $\sum_{n=0}^{\infty} \frac{(n+1)x^n}{(n+2)(n+3)}$;

(b) $\sum_{n=0}^{\infty} \frac{nx^n}{(n+1)^2}$;

(c) $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$;

(d) $\sum_{n=0}^{\infty} \frac{(n!)^2 x^{2n}}{(2n)!}$;

(e) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$

(f) $x + \frac{x^2}{2^2} + \frac{2!}{3^3}x^3 + \frac{3!}{4^4}x^4 + \cdots$

(3) If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then prove that, for any positive integer k, $\sum_{n=0}^{\infty} a_n x^{kn}$ has radius of convergence $R^{1/k}$.

(4) Let $f(x) = \exp^{-\frac{1}{x^2}}$ when $x \neq 0$ and $f(0) = 0$. Show that

(a) $f'(0) = 0$.

(b) for $x \neq 0$, $n \geq 1$, $f^{(n)}(x) = P_n(\frac{1}{x}) \exp^{-\frac{1}{x^2}}$, where P_n is a polynomial of degree $3n$.

(c) $f^{(n)}(0) = 0$ for $n = 1, 2, \ldots$

(d) the Maclaurin series of f converges to $f(x)$ only when $x = 0$.

(5) Show that

(a)

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, -1 \leq x \leq 1$$

(b)

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, -1 < x \leq 1$$