CONVERGENCE TESTS I: COMPARISON, LIMIT COMPARISON AND CAUCHY CONDENSATION TESTS

(1) Let $a_n \geq 0$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} a_n$ converges, then show that the following series also converge.

(a) $\sum_{n=1}^{\infty} a_n^2$.

(b) $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$.

(c) $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$.

(d) $\sum_{n=1}^{\infty} \frac{a_n+4^n}{a_n+5^n}$.

(2) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Show that $\sum_{n=1}^{\infty} |a_n|$ diverges if $\sum_{n=1}^{\infty} a_n^2$ diverges.

(3) Let $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges if and only if $\sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}$ converges.

(4) Suppose $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty} (1 - \frac{\sin a_n}{a_n})$ converges.

(5) Let (a_n) be a sequence of positive real numbers such that $a_{n+1} \leq a_n$ for all n and $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty} n(a_n - a_{n+1})$ converges.

(6) Show that $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))}$ diverges.

(7) In each of the following cases, discuss the convergence/divergence of the series $\sum_{n=2}^{\infty} a_n$ where a_n equals

a) $\frac{1}{(\ln n)^p}$, $(p > 0)$ b) $\frac{\sin \frac{1}{n}}{\sqrt{n}}$ c) $\frac{1}{n^2 - \ln n}$ d) e^{-n^2}

e) $\frac{1}{n^{1+\frac{1}{n}}}$ f) $1 - \cos \frac{\pi}{n}$ g) $(\ln n) \sin \frac{1}{n^2}$ h) $(n+2)(1-\cos \frac{1}{n})$

i) $\frac{3+\cos n}{e^n}$ j) $\frac{2+\sin^3(n+1)}{2^n+n^2}$ k) $\frac{\sqrt{n+1}-\sqrt{n}}{n}$