PROBLEM SET 01: THE REAL NUMBER SYSTEM

(1) Let A be a nonempty subset of $\mathbb{R}$ and $M \in \mathbb{R}$. Prove that $M = \sup A$ if and only if

   (a) M is an upper bound of A,

   (b) for any $\varepsilon > 0$, there exists $x \in A$ such that $x > M - \varepsilon$.

(2) Let A, B be nonempty subsets of $\mathbb{R}$ with $A \subset B$. Prove

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

(3) Prove that for any $x \in \mathbb{R}$, there exists $m \in \mathbb{N}$ such that $-m < x$.

(4) Let $x, y \in \mathbb{R}$ such that $x < y$. Show that there exist $m, n \in \mathbb{N}$ such that $x < x + \frac{1}{m} < y$ and $x < y - \frac{1}{n} < y$.

(5) (a) Let $x > 0$. Prove that there exists $n \in \mathbb{N}$ such that $x > \frac{1}{n}$.

   (b) Let $x \geq 0$. Prove that $x = 0$ if and only if $x \leq \frac{1}{n}$ for every $n \in \mathbb{N}$.

(6) Let $U_n = (0, \frac{1}{n})$ and $V_n = (\frac{1}{n}, 1)$. Find $\cap_n U_n$ and $\cup_n V_n$.

(7) Find the supremum and infimum of the following sets:

   (a) (a, b), where $a, b \in \mathbb{R}$.

   (b) $\{1 - \frac{1}{n^2} : n \in \mathbb{N}\}$.

   (c) $\{\frac{m+n}{mn} : m,\ n \in \mathbb{N}\}$.

   (d) $\{x \in \mathbb{R} : x^2 - 5x + 6 < 0\}$.

   (e) The set of real numbers in $(0, 1)$ whose decimal expansions contains only 0's and 1's.

(8) Let A be a nonempty subset of $\mathbb{R}$ and $x, M \in \mathbb{R}$. Define the distance between x and A by

$$d(x, A) = \inf\{|x - a| : a \in A\}.$$

If $M = \sup A$, show that $d(M, A) = 0$.

(9) Let $A, B \subset \mathbb{R}$ be nonempty such that $\alpha = \sup A$ and $\beta = \sup B$. Show that $A + B$ is bounded above and $\sup(A + B) = \alpha + \beta$.

(10) **($\mathbb{Q}$ does not have the LUB property)**

   (a) Let $x \in \mathbb{Q}$ and $x > 0$. If $x^2 < 2$, show that there exists $n \in \mathbb{N}$ such that $(x+\frac{1}{n})^2 < 2$. Likewise, if $x^2 > 2$, show that there exists $m \in \mathbb{N}$ such that $(x - \frac{1}{m})^2 > 2$.

   (b) Show that the set $A = \{r \in \mathbb{Q} : r > 0, r^2 < 2\}$ is bounded above in $\mathbb{Q}$ but it does not have the LUB property in $\mathbb{Q}$.

   (c) From part (b), $\mathbb{Q}$ does not have the LUB property.

   (d) Let A be the set defined in part (b) and $M = \sup A$. Show that $M^2 = 2$.