Problem Set 08: Fixed point iteration method and Newton's method

(1) Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable and $\alpha \in \mathbb{R}$ be such that $|g'(x)| \leq \alpha < 1$ for all $x \in \mathbb{R}$.

  (a) Show that the sequence generated by the fixed point iteration method for g converges to a fixed point of g for any starting value $x_0 \in \mathbb{R}$.

  (b) Show that g has a unique fixed point.

(2) Let $x_0 \in \mathbb{R}$. Using the fixed point iteration method generate a sequence of approximate solutions of the equation $x - \frac{1}{2}\sin x = 1$ for the starting value x_0.

(3) Let $g : [0, 1] \longrightarrow [0, 1]$ be defined by $g(x) = \frac{1}{1+x^2}$. Let (x_n) be the sequence generated by the fixed point iteration method for g with the starting value $x_0 = 1$. Show that (x_n) converges.

(4) Let $f(x) = e^{-\frac{1}{x^2}}$ if $x \neq 0$ and $f(0) = 0$. Suppose that $0 < x_0 < 1$ and (x_n) be the sequence generated by Newton's method with the starting value x_0. Show that (x_n) converges.

(5) Let $f(x) = 3x^{1/3}$. Let $x_0 > 0$ and (x_n) be the sequence generated by Newton's method. Show that (x_n) is unbounded.